Spectral decomposition of some non-self-adjoint operators

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Introduction

Some motivations (I)

Nuclear optical model

- Feshbach, Porter and Weisskopf ('54): nuclear optical model describing both elastic scattering and absorption of a neutron targeted onto a nucleus
- "Pseudo-Hamiltonian" on $L^2(\mathbb{R}^3)$

$$H = -\Delta + V(x) - iW(x)$$

  with V and W real-valued, bounded and compactly supported, $W \geq 0$
- $-iH$ generates a strongly continuous semigroup of contractions. Dynamics described by the Schrödinger equation

$$\begin{cases} i\partial_t u_t = Hu_t \\ u_0 \in \mathcal{D}(H) \end{cases}$$

- Probability that the neutron, initially in the normalized state u_0 (supposed to be orthogonal to bound states), eventually escapes from the nucleus:

$$p_{\mathrm{scatt}}(u_0) = \lim_{t\to\infty} \left\| e^{-itH} u_0 \right\|^2$$

- Probability of absorption:

$$p_{\mathrm{abs}}(u_0) = 1 - \lim_{t\to\infty} \left\| e^{-itH} u_0 \right\|^2$$

- Empirical model widely used in Nuclear Physics

Some motivations (II)

$\mathcal{PT}$-symmetric operators

- [Bender, Boettcher '98]: large class of '$\mathcal{PT}$-invariant Hamiltonians' have real spectra
- For Schrödinger operators $H = -\Delta + V(x)$ on $L^2(\mathbb{R}^d)$, $\mathcal{PT}$-symmetry means that
$$\overline{V}(-x) = V(x)$$
- [Borisov, Krejcirik '08, '12], [Wen, Bender '20]: examples of $\mathcal{PT}$-symmetric Schrödinger operators having continuous spectra

Non-self-adjoint operators in Quantum Mechanics

- Holomorphic families of closed operators [Dereziński and collaborators]
- [Bagarello, Gazeau, Szafraniec, Znojil '15]: *Non-Selfadjoint Operators in Quantum Physics. Mathematical Aspects.*
- [Krejcirik '17]: *Mathematical aspects of quantum mechanics with non-self-adjoint operators.*

Abstract framework

Abstract model

The model

- $\mathcal{H}$ complex Hilbert space
- Hamiltonian
$$H = H_0 + V = H_0 + CWC,$$
with $H_0 \geq 0$, $C \in \mathcal{B}(\mathcal{H})$, $C > 0$ and relatively compact with respect to H_0, $W \in \mathcal{B}(\mathcal{H})$ arbitrary
- H is a closed operator with domain
$$\mathcal{D}(H) = \mathcal{D}(H_0)$$
- $-iH$ generates a strongly continuous group $\{e^{-itH}\}_{t\in\mathbb{R}}$ s.t.
$$\left\| e^{-itH} \right\| \leq e^{\|V\||t|}, \ t \in \mathbb{R}$$
- $H^* = H_0 + CW^*C$ with domain $\mathcal{D}(H^*) = \mathcal{D}(H_0)$
- $\sigma_{\mathrm{ess}}(H) = \sigma_{\mathrm{ess}}(H_0)$ and $\sigma(H) \setminus \sigma_{\mathrm{ess}}(H)$ consists of an at most countable number of eigenvalues of finite algebraic multiplicities that can only accumulate at points of $\sigma_{\mathrm{ess}}(H)$

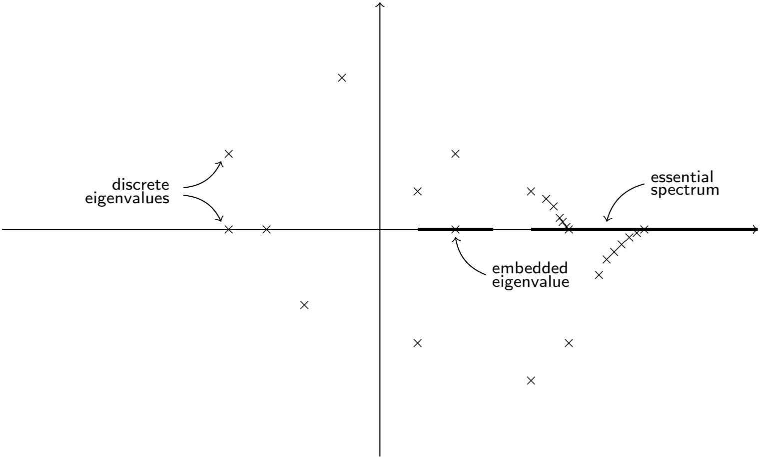


FIGURE: **Spectrum of H**

Point spectral subspace (I)

Point spectrum

$$\sigma_{\mathrm{p}}(H) := \{\lambda \in \mathbb{C},\ \mathrm{Ker}(H-\lambda) \neq \{0\}\}$$

Algebraic multiplicity of an eigenvalue $\lambda \in \sigma_{\mathrm{p}}(H)$:

$$\mathrm{m}_\lambda := \dim\Big(\bigcup_{k\geq 1} \mathrm{Ker}\big((H-\lambda)^k\big)\Big)$$

Discrete spectrum and discrete spectral subspace

$$\sigma_{\mathrm{disc}}(H) := \sigma(H) \setminus \sigma_{\mathrm{ess}}(H) \subset \sigma_{\mathrm{p}}(H)$$

- For $\lambda \in \sigma_{\mathrm{disc}}(H)$, Riesz projection defined by

$$\Pi_\lambda = -\frac{1}{2i\pi}\int_\gamma R_H(z)dz, \quad R_H(z) = (H-z)^{-1},$$

  where γ is a circle centered at λ, of sufficiently small radius
- $\mathrm{Ran}(\Pi_\lambda)$ spanned by generalized eigenvectors of H associated to λ, $u \in \mathcal{D}(H^k)$ s.t. $(H-\lambda)^k u = 0$
- Discrete spectral subspace:

$$\mathcal{H}_{\mathrm{disc}}(H) = \mathrm{Span}\left\{u \in \mathrm{Ran}(\Pi_\lambda),\ \lambda \in \sigma_{\mathrm{disc}}(H)\right\}^{\mathrm{cl}}$$

Point spectral subspace (II)

Set of embedded eigenvalues

$$\sigma_{\mathrm{emb}}(H) := \sigma_{\mathrm{p}}(H) \cap \sigma_{\mathrm{ess}}(H)$$

To define spectral projections corresponding to embedded eigenvalues

- We suppose the existence of a conjugation operator $J \in \mathcal{B}(\mathcal{H})$ satisfying

$$J\mathcal{D}(H_0) \subset \mathcal{D}(H_0) \quad \text{and} \quad \forall u \in \mathcal{D}(H_0), \quad JHu = H^* Ju$$

- If $\lambda \in \sigma_{\mathrm{emb}}(H)$, we suppose that $\mathrm{m}_\lambda < \infty$ and that the symmetric bilinear form

$$\mathrm{Ker}((H-\lambda)^{\mathrm{m}_\lambda}) \ni (u,v) \mapsto \langle Ju, v \rangle \quad \text{is non-degenerate}$$

- Under these conditions, there exists a basis $(\varphi_k)_{1 \le k \le \mathrm{m}_\lambda}$ of $\mathrm{Ker}((H-\lambda)^{\mathrm{m}_\lambda})$ such that $\langle J\varphi_i, \varphi_j \rangle = \delta_{ij}$, $1 \le i,j \le \mathrm{m}_\lambda$. Then

$$\Pi_\lambda u := \sum_{k=1}^{\mathrm{m}_\lambda} \langle J\varphi_k, u \rangle \varphi_k, \quad u \in \mathcal{H}$$

- $$\mathcal{H}_{\mathrm{emb}}(H) := \mathrm{Span}\,\{u \in \mathrm{Ran}(\Pi_\lambda),\ \lambda \in \sigma_{\mathrm{emb}}(H)\}^{\mathrm{cl}}$$

Point spectral subspace (III)

Point spectral subspace

$$\mathcal{H}_{\mathrm{p}}(H) = \mathcal{H}_{\mathrm{disc}}(H) \oplus \mathcal{H}_{\mathrm{emb}}(H)$$

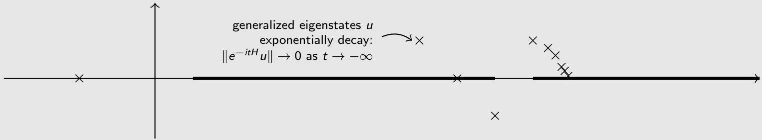
Asymptotically disappearing states

Subspaces of asymptotically disappearing states

$$\mathcal{H}^{\pm}_{\mathrm{ads}}(H) := \Big\{ u \in \mathcal{H},\ \lim_{t \to \pm\infty} \| e^{-itH} u \| = 0 \Big\}^{\mathrm{cl}}$$

Relation with discrete generalized eigenstates

- Easy to see that
  $\mathrm{Span}\,\{\text{Gen. eigenstates associated to } \lambda, \pm\mathrm{Im}(\lambda) < 0\}^{\mathrm{cl}} \subset \mathcal{H}^{\pm}_{\mathrm{ads}}(H)$



- Question: conditions implying that the previous inclusion becomes an equality?
- [Kato '66] For small perturbations, H and H_0 are similar, hence $\mathcal{H}^{\pm}_{\mathrm{ads}}(H) = \{0\}$
- For dissipative operators, $\mathrm{Im}(V) \leq 0$, the question was left as an open problem in [Davies '80], with an answer given in [F., Fröhlich '18]

Absolutely continuous spectral subspace

Absolutely continuous spectral subspace

$$\mathcal{H}_{\mathrm{ac}}(H) := \Big\{ u \in \mathcal{H},\ \exists c_u > 0, \forall v \in \mathcal{H},\ \int_{\mathbb{R}} \big| \langle e^{-itH} u, v \rangle \big|^2 \mathrm{d}t \le c_u \|v\|^2 \Big\}^{\mathrm{cl}}$$

Relation with point spectral subspace of H^*

- Not difficult to verify that

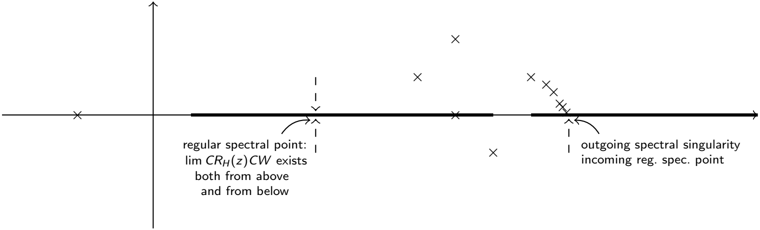
$$\mathcal{H}_{\mathrm{ac}}(H) \subset \mathcal{H}_{\mathrm{p}}(H^*)^{\perp}$$

- Question: conditions implying that the previous inclusion becomes an equality?
- Other definitions considered in the literature: [Davies '79] for dissipative operators, [Naboko '76] using the theory of dilations of dissipative operators
- Under suitable assumption, coincides with the space of 'scattering states'

Spectral singularities (I)

Definition: Spectral singularity

1. $\lambda \in \sigma_{\mathrm{ess}}(H)$ is an outgoing/incoming regular spectral point of H if λ is not an accumulation point of eigenvalues located in $\lambda \pm i\,(0,\infty)$ and if the limit
$$CR_H(\lambda \pm i0^+)CW := \lim_{\varepsilon \to 0^+} CR_H(\lambda \pm i\varepsilon)CW$$
exists in the norm topology of $\mathcal{B}(\mathcal{H})$
2. λ is a regular spectral point of H if it is both an incoming and an outgoing regular spectral point of H
3. Spectral singularity = not regular spectral point



Spectral singularities (II)

Remarks

For dissipative operators, similar definition in [F, Fröhlich '18]. Other related notions:

- [Dunford '58] (theory of spectral operators), [Schwartz '59] (spectral singularity = singular point of a 'spectral resolution' for non-self-adjoint operators)
- [F, Nicoleau '19] (for dissipative operators, spectral singularity = point of the essential spectrum where the scattering matrix is non-invertible)
- For one-dimensional Schrödinger operators (spectral singularity = zero of the Jost function)

Some properties [F, Frantz]

- λ embedded eigenvalue $\Rightarrow$ λ both outgoing and incoming spectral singularity
- At thresholds, outgoing and incoming spectral singularities coincide

Spectral singularities (III)

Proposition (Schrödinger operators in 3-dimension)

Suppose that V is a complex-valued potential such that $\langle x\rangle^{\sigma} V(x) \in L^{\infty}(\mathbb{R}^3)$ with $\sigma > 1$. Let $C(x) = \langle x\rangle^{-\sigma/2}$. Then for all $\lambda > 0$, the following conditions are equivalent:

1. λ is an outgoing/incoming spectral singularity of $H = -\Delta + V$
2. There exists $\Psi \neq 0$, $\langle x\rangle^{-\sigma/2}\Psi \in L^2$, Ψ satisfying the outgoing/incoming Sommerfeld radiation condition, such that

$$(-\Delta + V(x) - \lambda)\Psi = 0$$

The same holds at the threshold $\lambda = 0$ if $\langle x\rangle^{\sigma} V(x) \in L^{\infty}(\mathbb{R}^3)$ with $\sigma > 2$

Remarks

- There is an abstract version of this proposition involving the Gelfand triple

$$\mathrm{Ran}(C) \hookrightarrow \mathcal{H} \hookrightarrow (\mathrm{Ran}(C))'.$$

- [Wang '12]: For any $\lambda > 0$, one can construct a smooth compactly supported potential V such that λ is a spectral singularity of H

Spectral decomposition

Hypotheses (I)

(H1) Limiting absorption principle for H_0

$$\sup_{\pm \mathrm{Im}(z)>0} \left\| C R_0(z) C \right\| < \infty$$

Remark

Satisfied for $H_0 = -\Delta$, $C(x) = \langle x \rangle^{-\sigma/2}$, $\sigma > 2$ in dimension $d \geq 3$

Consequences

- The spectrum of H_0 is purely absolutely continuous, i.e.

$$\sigma_{\mathrm{pp}}(H_0) = \emptyset, \quad \sigma_{\mathrm{ac}}(H_0) = \sigma(H_0), \quad \sigma_{\mathrm{sc}}(H_0) = \emptyset$$

- The limits $C R_0(\lambda \pm i0^+) C$ exist for almost every $\lambda \in \sigma_{\mathrm{ess}}(H)$, in the norm topology of $\mathcal{B}(\mathcal{H})$

Hypotheses (II)

(H2) Eigenvalues of H

H only has a finite number of eigenvalues with finite algebraic multiplicities

Remark

- Satisfied for Schrödinger operators $H = -\Delta + V(x)$ in $L^2(\mathbb{R}^3)$ if V is exponentially decaying [Frank, Laptev, Safronov '16]
- Does not exclude embedded eigenvalues

Hypotheses (III)

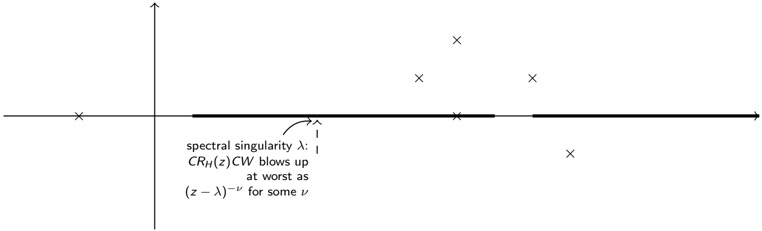
(H3) Spectral singularities for H

H only has a finite number of spectral singularities $\{\lambda_1, \dots \lambda_n\} \subset \sigma_{\mathrm{ess}}(H)$ and there exist $\varepsilon_0 > 0$ and integers $\nu_1, \dots, \nu_n, \nu_\infty \geq 0$ such that

$$\sup_{\mathrm{Re}(z)\in\sigma_{\mathrm{ess}}(H),\pm\mathrm{Im}(z)\in(0,\varepsilon_0)} |r(z)| \| CR_H(z)CW \| < \infty,$$

where z_0 is an arbitrary complex number such that $z_0 \in \rho(H)$, $z_0 \in \mathbb{C} \setminus \mathbb{R}$ and

$$r(z) := \frac{1}{(z-z_0)^{\nu_\infty}} \prod_{j=1}^{n} \frac{(z-\lambda_j)^{\nu_j}}{(z-z_0)^{\nu_j}}$$



Hypotheses (IV)

Remarks

- The factors $(z-\lambda_j)^{\nu_j}$ 'regularize' the singularities of $z \mapsto CR_H(z)CW$ as z approaches λ_j. Dividing them by $(z-z_0)^{\nu_j}$ produces bounded terms
- The factor $\frac{1}{(z-z_0)^{\nu_\infty}}$ 'regularize' a possible singularity at ∞
- For Schrödinger operators $H = -\Delta + V(x)$ in $L^2(\mathbb{R}^3)$ with V compactly supported, (H3) is satisfied with ν_j the multiplicity of the resonance λ_j and $\nu_\infty = 0$

Asymptotically disappearing states

Recall that

$$\mathcal{H}_{\mathrm{ads}}^{\pm}(H) := \left\{ u \in \mathcal{H}, \lim_{t \to \pm\infty} \|e^{-itH} u\| = 0 \right\}^{\mathrm{cl}}$$

Theorem [F, Frantz]

Suppose that Hypotheses (H1)–(H3) hold. Then

$$\mathcal{H}_{\mathrm{ads}}^{\pm}(H) = \mathrm{Span}\left\{\text{Gen. eigenstates associated to } \lambda, \pm\mathrm{Im}(\lambda) < 0\right\}^{\mathrm{cl}}$$

Remark

For dissipative operators, analogous result proven in [F. Fröhlich '18]. The proof in [F. Fröhlich '18] relies on the existence and properties of wave operators. Our proof does not rely on scattering theory

Theorem (Consequence for Schrödinger operators)

Suppose that V is a complex-valued potential such that $V \in L^\infty_{\mathrm{c}}(\mathbb{R}^3)$. Then the previous theorem applies to $H = -\Delta + V$.

Hypotheses (IV)

(H4) Conjugation operator

There exists an anti-linear continuous map $J : \mathcal{H} \to \mathcal{H}$ such that

1. $J\mathcal{D}(H_0) \subset \mathcal{D}(H_0)$ and $\forall u \in \mathcal{D}(H_0)$, $JH_0 u = H_0 Ju$
2. $JC = CJ$ and $JW = W^* J$

Moreover, for all embedded eigenvalues $\lambda \in \sigma_{\mathrm{ess}}(H)$, the symmetric bilinear form

$$\mathrm{Ker}\big((H-\lambda)^{\mathrm{m}_\lambda}\big) \ni (u,v) \mapsto \langle Ju, v\rangle \quad \text{is non-degenerate}$$

Remark

For Schrödinger operators $H = -\Delta + V(x)$, J is the complex conjugation and Hypothesis (H4) means that for all embedded eigenvalues $\lambda \in [0,\infty)$,

$$\mathrm{Ker}\big((H-\lambda)^{\mathrm{m}_\lambda}\big) \ni (u,v) \mapsto \int_{\mathbb{R}^3} u(x)v(x)\mathrm{d}x \quad \text{is non-degenerate}$$

Absolutely continuous spectral subspace

Recall that

$$\mathcal{H}_{\mathrm{ac}}(H) := \left\{ u \in \mathcal{H},\ \exists c_u > 0, \forall v \in \mathcal{H},\ \int_{\mathbb{R}} \left| \langle e^{-itH} u, v \rangle \right|^2 \mathrm{d}t \leq c_u \|v\|^2 \right\}^{\mathrm{cl}}$$

Theorem [F, Frantz]

Suppose that Hypotheses (H1)–(H3) hold. If H has embedded eigenvalues, suppose in addition that (H4) holds. Then

$$\mathcal{H}_{\mathrm{ac}}(H) = \mathcal{H}_{\mathrm{p}}(H^*)^{\perp}$$

Remark: comparable results in the literature (only for H dissipative)

[Simon '79] for dissipative Schrödinger operators, [Davies '80] for abstract dissipative operators using the theory of dilations, with a different definition of $\mathcal{H}_{\mathrm{ac}}$ and a different result ($\mathcal{H}_{\mathrm{ac}}$ coincides with the orthogonal complement of 'bound states')

Theorem (Consequence for Schrödinger operators)

Suppose that V is a complex-valued potential such that $V \in L^\infty_{\mathrm{c}}(\mathbb{R}^3)$ and the previous hypothesis on embedded eigenvalues is satisfied. Then the previous theorem applies to $H = -\Delta + V$

Spectral decomposition

Consequence of the previous two theorems

Suppose that Hypotheses (H1)–(H4) hold. Then we have the following J-orthogonal direct sum decompositions of the Hilbert space:

$$\begin{aligned}\mathcal{H} &= \mathcal{H}_{\rm ac}(H) \oplus \mathcal{H}_{\rm p}(H)\\ &= \mathcal{H}_{\rm ac}(H) \oplus \mathcal{H}_{\rm disc}(H) \oplus \mathcal{H}_{\rm emb}(H)\\ &= \mathcal{H}_{\rm ac}(H) \oplus \mathcal{H}^{+}_{\rm ads}(H) \oplus \mathcal{H}^{-}_{\rm ads}(H) \oplus \mathcal{H}_{\rm b}(H),\end{aligned}$$

where $\mathcal{H}_{\rm b}(H)$ is the space of ‘bound states’, i.e. the closure of the vector space spanned by all generalized eigenvectors of H corresponding to real eigenvalues (either isolated or embedded)

Regularized functional calculus

Regularized Stone formula

Stone's formula for self-adjoint operators

Suppose H is a self-adjoint operator without embedded eigenvalues. Then

$$\mathrm{Id} = \sum_{\lambda \in \sigma_{\mathrm{disc}}(H)} \Pi_\lambda + \operatorname*{w-lim}_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int_{\sigma_{\mathrm{ess}}(H)} \big(R_H(\lambda + i\varepsilon) - R_H(\lambda - i\varepsilon) \big) \mathrm{d}\lambda$$

Regularized version

Recall that we have assumed

$$\sup_{\substack{\mathrm{Re}(z) \in \sigma_{\mathrm{ess}}(H) \\ \pm \mathrm{Im}(z) \in (0, \varepsilon_0)}} |r(z)| \big\| C R_H(z) C W \big\| < \infty, \quad r(z) = \frac{1}{(z - z_0)^{\nu_\infty}} \prod_{j=1}^{n} \frac{(z - \lambda_j)^{\nu_j}}{(z - z_0)^{\nu_j}}$$

Then, under our assumptions, we have

$$r(H) = \sum_{\lambda \in \sigma_{\mathrm{disc}}(H)} r(H) \Pi_\lambda + \operatorname*{w-lim}_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int_{\sigma_{\mathrm{ess}}(H)} r(\lambda) \big(R_H(\lambda + i\varepsilon) - R_H(\lambda - i\varepsilon) \big) \mathrm{d}\lambda$$

Idea of the proof

Riesz-Dunford functional calculus

$$r(H) = -\frac{1}{2i\pi}\int_{\Gamma_\varepsilon} r(z)R_H(z)\mathrm{d}z \quad \text{then} \quad \varepsilon \to 0^+$$

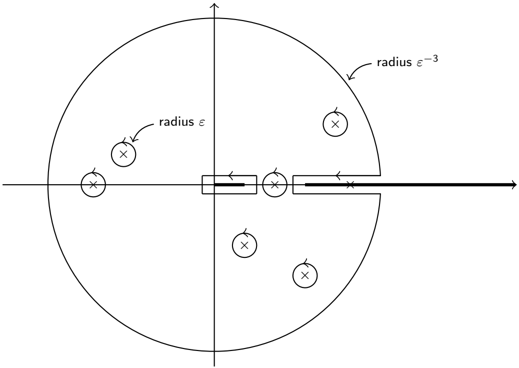


FIGURE: **The contour Γ_ε.**

Functional calculus (I)

Recall Hypothesis (H1):

$$\sup_{\pm \mathrm{Im}(z)>0} \left\| C R_0(z) C \right\| < \infty$$

Proposition (Functional calculus in intervals not containing spectral singularities)

Suppose (H1). Let $I \subset \mathbb{R}$ be a closed interval and suppose that there exists $\varepsilon_0 > 0$ such that

$$\sup_{\mathrm{Re}(z)\in I, \pm\mathrm{Im}(z)\in(0,\varepsilon_0)} \left\| C R_H(z) C W \right\| < \infty.$$

Then the map

$$\mathrm{C_b}(I) \ni f \mapsto f(H) := \operatorname*{w-lim}_{\varepsilon\to 0^+} \frac{1}{2\pi i} \int_I f(\lambda) \big(R_H(\lambda + i\varepsilon) - R_H(\lambda - i\varepsilon) \big) \mathrm{d}\lambda \in \mathcal{B}(\mathcal{H})$$

Remark

Related to the Dunford-Schwartz theory of spectral operators

Functional calculus (II)

Proposition (Regularized functional calculus)

Suppose (H1). Let $I \subset \mathbb{R}$ be a closed interval and suppose that there exists $\varepsilon_0 > 0$ and a bounded holomorphic function h such that

$$\sup_{\substack{\mathrm{Re}(z)\in I \\ \pm\mathrm{Im}(z)\in(0,\varepsilon_0)}} |h(z)| \big\| C R_H(z) C W \big\| < \infty, \qquad \lambda \mapsto \sup_{0<\varepsilon<\varepsilon_0} |h'(\lambda \pm i\varepsilon)| \in L^2(I).$$

Then the map

$$\mathrm{C}_{\mathrm{b,reg}}(I) \ni f \mapsto f(H) := \text{w-lim}_{\varepsilon\to 0^+} \frac{1}{2\pi i} \int_I g(\lambda) \big(h(\lambda+i\varepsilon) R_H(\lambda+i\varepsilon) - h(\lambda-i\varepsilon) R_H(\lambda-i\varepsilon) \big) \mathrm{d}\lambda \in \mathcal{B}(\mathcal{H})$$

is an algebra morphism and there exists $\mathrm{c} > 0$ such that

$$\|f(H)\|_{\mathcal{B}(\mathcal{H})} \le \mathrm{c} \|g\|_{L^\infty}.$$

Here

$$\mathrm{C}_{\mathrm{b,reg}}(I) := \big\{ f : I \to \mathbb{C},\ \exists g \in \mathrm{C}_{\mathrm{b}}(I),\ f = hg \big\}$$

Remark

Other functional calculi for operators on Banach spaces under an assumption of polynomial growth of the resolvent near the real axis: [Davies '95] (general theory), [Georgescu, Gérard, Häfner '13] (Krein spaces)

Thank you !

Spectral singularities and resonances (I)

$\mathcal{H} = L^2(\mathbb{R}^d)$, $H_0 = -\Delta$, V compactly supported

Resonance may be defined as a pole of the map

$$\mathbb{C} \ni z \mapsto (H - z^2)^{-1} : L^2_{\mathrm{c}}(\mathbb{R}^3) \to L^2_{\mathrm{loc}}(\mathbb{R}^3),$$

Then

$$\text{Spectral singularity at } \lambda > 0 = \text{resonance at } \pm\lambda^{1/2}$$

Remarks

- Resonances theory: [Sjöstrand '02], [Dyatlov-Zworski '18]
- [Wang '12]: For any $\lambda > 0$, one can construct a smooth compactly supported potential V such that λ is a spectral singularity of H

Spectral singularities and resonances (II)

Example: $\mathcal{H} = L^2(\mathbb{R}^3)$, $H_0 = -\Delta$, V short-range

If $V(x) = \mathcal{O}(\langle x \rangle^{-\delta})$ with $\delta > 1$, then $\pm\lambda^{1/2}$ (with $\lambda > 0$) may be called a resonance of H if the equation $(H - \lambda)u = 0$ admits a distributional solution (called a resonant state)

$$u \in \bigcap_{\sigma > 1} L^2_{-\sigma/2} \setminus L^2$$

satisfying the Sommerfeld radiation condition

$$u(x) = |x|^{-1} e^{\pm i \lambda^{\frac{1}{2}} |x|} \Big(a(\frac{x}{|x|}) + o(1) \Big), \quad |x| \to \infty,$$

with $a \in L^2(S^2)$, $a \neq 0$. Here $L^2_{-\sigma/2} = \big\{ f : \mathbb{R}^3 \to \mathbb{C},\, x \mapsto \langle x \rangle^{-\frac{\sigma}{2}} f(x) \in L^2(\mathbb{R}^3) \big\}$

Spectral singularities: Characterization (I)

Assumption (given $\lambda \in \sigma_{\text{ess}}(H)$, with $R_0 := R_{H_0}$)

$$CR_0(\lambda \pm i0^+)C := \lim_{\varepsilon \to 0^+} CR_0(\lambda \pm i\varepsilon)C \text{ exist in the topology of } \mathcal{B}(\mathcal{H}) \qquad (*)$$

Free Schrödinger operator in $\mathcal{H} = L^2(\mathbb{R}^3)$

- For $\lambda > 0$, the limits
$$\langle x \rangle^{-s}\big(- \Delta - (\lambda \pm i0^+)\big)\langle x \rangle^{-s},$$
exist in the norm topology of $\mathcal{B}(\mathcal{H})$, for any $s > \frac{1}{2}$, where $\langle x \rangle := (1 + x^2)^{\frac{1}{2}}$
- If $\lambda = 0$, the limits
$$\langle x \rangle^{-s}(-\Delta \pm i0^+)\langle x \rangle^{-s},$$
exist (and coincide) for any $s > 1$

Spectral singularities: Characterization (II)

Extension of the Hilbert space

- Let $\mathcal{H}_C := \mathrm{Ran}(C)$ equipped with $\langle u, v\rangle_{\mathcal{H}_C} := \langle C^{-1}u, C^{-1}v\rangle$
- Let $\mathcal{H}'_C$ be the anti-dual of $\mathcal{H}_C$. We obtain the Gelfand triple

$$\mathcal{H}_C \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}'_C$$

- Assuming that $\mathcal{D}(H_0|_{\mathcal{H}_C}) := \{u \in \mathcal{D}(H_0) \cap \mathcal{H}_C, H_0 u \in \mathcal{H}_C\}$ is dense in $\mathcal{H}_C$, H extends to

$$H' = H'_0 + CWC' : \mathcal{H}'_C \to \mathcal{H}'_C$$

- $CR_0(\lambda \pm i0^+)C := \lim_{\varepsilon\to 0^+} CR_0(\lambda \pm i\varepsilon)C$ exists in $\mathcal{B}(\mathcal{H})$ is equivalent to

$$R_0(\lambda \pm i0^+) := \lim_{\varepsilon\to 0^+} R_0(\lambda \pm i\varepsilon) \text{ exist in } \mathcal{B}(\mathcal{H}_C, \mathcal{H}'_C)$$

Incoming/outgoing resonant states

Let $\lambda \in \sigma_{\mathrm{ess}}(H)$ be a spectral singularity of H. The space $\mathcal{H}'^{\pm}_C(\lambda) \subset \mathcal{H}'_C$ of outgoing/incoming resonant states corresponding to λ is defined by

$$\mathcal{H}'^{\pm}_C(\lambda) := \mathrm{Ker}\left(\mathrm{Id} + R_0(\lambda \pm i0^+)CWC'\right)$$

Spectral singularities: Characterization (III)

Theorem [F, Frantz]

Suppose that $(*)$ holds. The following conditions are equivalent:

1. λ is an outgoing/incoming spectral singularity of H
2. λ is an eigenvalue of H' associated to an eigenvector $\Psi \in \mathcal{H}_C'^{\pm}(\lambda)$

Some consequences

- λ embedded eigenvalue $\Rightarrow$ λ both outgoing and incoming spectral singularity
- At thresholds, outgoing and incoming spectral singularities coincide
- Suppose that V is a complex-valued potential such that $\langle x \rangle^{\sigma} V(x) \in L^{\infty}(\mathbb{R}^3)$ with $\sigma > 1$. Let $C(x) = \langle x \rangle^{-\sigma/2}$. Then for all $\lambda > 0$, the following conditions are equivalent:
    1. λ is an outgoing/incoming spectral singularity of H
    2. There exists $\Psi \in \mathcal{H}_C'^{\pm}(\lambda) \subset L^2_{-\sigma/2}$, $\Psi \neq 0$, such that

$$(-\Delta + V(x) - \lambda)\Psi = 0$$

- The same holds at the threshold $\lambda = 0$ if $\langle x \rangle^{\sigma} V(x) \in L^{\infty}(\mathbb{R}^3)$ with $\sigma > 2$

Spectral singularities: Ingredient of the proof

Proposition (Birman-Schwinger principle for spectral singularities)

Suppose that $(*)$ holds. Then the following conditions are equivalent:

1. λ is an outgoing/incoming regular spectral point of H
2. $\mathrm{Id} + CR_0(\lambda \pm i0^+)CW$ is invertible in $\mathcal{B}(\mathcal{H})$

Remark

Birman-Schwinger principle recently studied in abstract non-self-adjoint settings:

- [Behrndt, ter Elst, Gesztesy '20]
- [Hansmann, Krejcirik '20]