QNMs as eigenvalues of non-selfadjoint operators: the stability vs the definition problem

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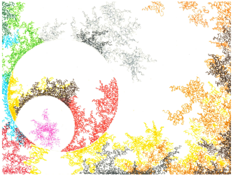

Spectral and Resonance Problems for Imaging, Seismology and Materials Science

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Scheme

1. The Problem in a nutshell: Black Hole Quasi-Normal Mode instability
2. The approach: spectral instability of QNMs as eigenvalues
3. QNMs as eigenvalues of non-selfadjoint operator: the "definition problem"
4. QNMs as eigenvalues of non-selfadjoint operator: the "stability problem"
5. The main (urgent and unsolved) problem in the approach
6. Conclusions and Perspectives

Scheme

BH quasi-normal modes as a spacetime probe

Black hole Quasi-Normal Modes (QNMs): a cornerstone in Gravitation

Resonant response of black holes under linear perturbations: complex frequencies providing an invariant **probe into the background spacetime geometry**.

i) Relativistic astrophysics and gravitational wave physics.

ii) Gravity and the quantum: semiclassical gravity, AdS/CFT-fluid/gravity dualities, analogue gravity...

iii) Mathematical relativity: stability of spacetimes.

iv) ... (Interdisciplinary physics)

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Are QNMs "structurally stable" under small perturbations?

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Are QNMs "structurally stable" under small perturbations?

An approach to the question: QNMs as eigenvalues [JLJ, R.P. Macedo, L. Al Sheikh 21, ...]

- Eigenvalues seem indeed unstable under high-frequency/low-regularity perturbations: *Can we measure the '(effective) regularity' of spacetime?*
- **Caveat (but!)**: **fundamental issues in** tools such as **Pseudospectrum**.

Black Hole QNM instabilities [cf. Nollert 96, Nollert & Price 99]

Perturbation theory on a Schwarzschild Black Hole: spherically symmetric case

Scalar, electromagnetic and gravitational perturbations reduced to (Minkowski) 1+1 wave equation for $\phi_{\ell m}(t, r_*)$ with a potential V_ℓ [Regge-Wheeler 57, Zerilli 70]:

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2} + V_\ell\right)\phi_{\ell m} = 0 \quad , \quad t \in]-\infty, \infty[\ , \ r^* \in]-\infty, \infty[$$

Schwarzschild quasi-normal modes

Convention for a "mode": $\phi_{\ell m}(t, r_*) \sim e^{i\omega t}\hat{\phi}_{\ell m}(r_*)$.
"Spectral" problem with **"outgoing boundary" conditions**:

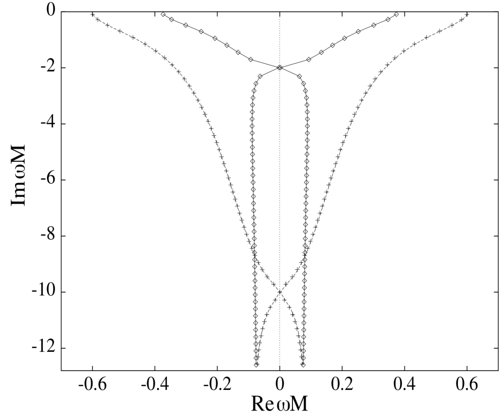
$$\left(-\frac{\partial^2}{\partial r_*^2} + V_\ell\right)\hat{\phi}_{\ell m} = \omega^2\hat{\phi}_{\ell m} \quad , \quad r^* \in]-\infty, \infty[$$

$$\hat{\phi}_{\ell m} \sim e^{-i\omega r_*}\ ,\ (r_* \to \infty) \quad , \quad \hat{\phi}_{\ell m} \sim e^{i\omega r_*}\ ,\ (r_* \to -\infty)$$

Time evolution stability: $\mathrm{Im}(\omega) > 0$. Exponential divergence of $\hat{\phi}_{\ell m}$ at $\pm\infty$.

Black Hole QNM instabilities [cf. Nollert 96, Nollert & Price 99]

Schwarzschild gravitational QNMs



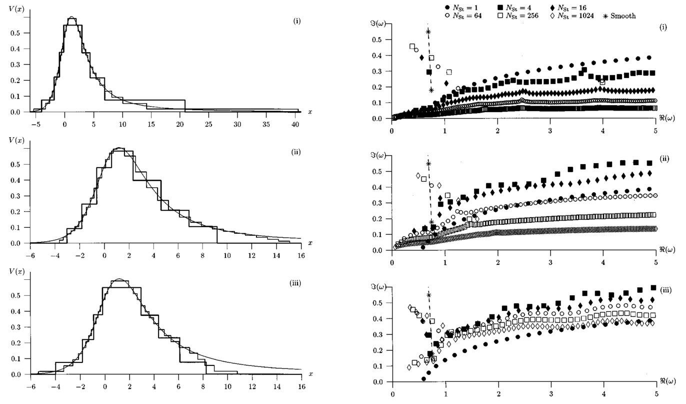
Schwarzschild QNMs ($\ell = 2$ diamonds, $\ell = 3$ crosses)

[e.g. Kokkotas & Schmidt 99, Berti, Cardoso & Starinets, 06, Konoplya & Zhidenko 11, ...]

QNM frequencies ω_n are invariant probes into the background geometry

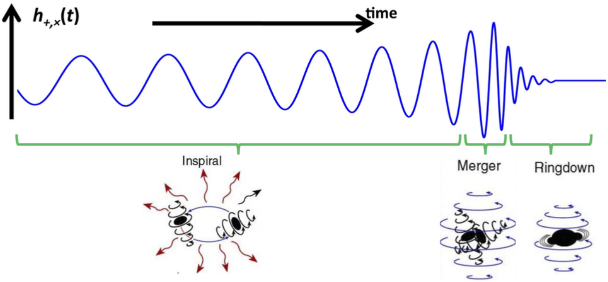
Black Hole QNM instabilities [cf. Nollert 96, Nollert & Price 99]

Nollert's work on stair-case discretizations of Schwarzschild (revisited in [Daghigh, Green & Morey 20])



- Instability of the slowest decaying QNM (but ringdown "stability").
- Instabilities of "highly damped QNMs".

Black Hole Spectroscopy: A Gravitational Wave Program



QNM frequencies ω_n as cornerstone of the program:

- Spacetime uniqueness theorem: Kerr QNM frequencies ω_n's.
- Ringdown as a resonant expansion in QNMs:

$$h(t) \sim \sum_n a_n e^{-\mathrm{Im}(\omega_n)t} \sin(\mathrm{Re}(\omega_n)t + \phi_n)$$

- QNM instability and reassessment of the interpretation scheme.
- QNM instability and inverse problem: Data Analysis issues.

Scheme

Posing the problem: eigenvalue perturbation theory

QNMs in the theory of Scattering Resonances: $P_V = -\Delta + V$

[Lax & Phillips, Vainberg; Sjöstrand, Zworski, Korotyaev, Iantchenko, Galkowski,...; e.g. Dyatlov & Zworski 20]

Resolvent $R_V(\lambda) = (P_V - \lambda)^{-1}$ analytic $\mathrm{Im}(\lambda) > 0$. Scattering resonances: poles of the meromorphic extension of (truncated) $R_V(\lambda)$ to $\mathrm{Im}(\lambda) < 0$.

QNMs as a "proper" eigenvalue problem: **non-selfadjoint operators**

- "Complex scaling" [Simon 78, Reed & Simon 78, Sjöstrand...]: not the approach here.
- **Hyperboloidal approach** [Friedman & Schutz 75, Schmidt 93, Bizon, Zenginoglu 11, Vasy 13, Warnick 15, Ansorg & P.-Macedo 16, Gajic & Warnick 19, Bizon et al. 20, Galkowski & Zworski 20, ...]
  Problem in terms of "eingenvalue problem" of **non-selfadjoint operator** L:
  $$\boxed{L\, u_{\ell m} = \omega\, u_{\ell m}} \quad , \quad u_{\ell m} \in H \text{ (Hilbert space)}$$
  - **Geometric boundary conditions**: Null infinity reached by hyperboloidal slices.
  - **Regularity conditions on** $u_{\ell m}$: choice of appropriate H, then $\omega \in \sigma_p(L)$.

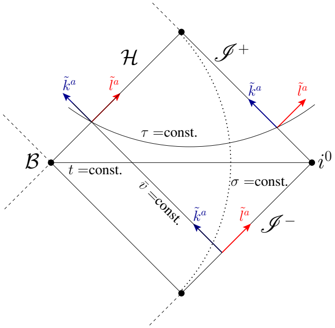
QNM instability: perspective from spectral theory of non-selfadjoint operators

- Spectral instability of non-selfadjoint (non-normal) operators.
- Adapted tools: e.g. **Pseudospectrum**.

Hyperboloidal slices: geometric outgoing BCs at $\mathscr{I}^+$

Hyperboloidal approach to QNMs

- **Spectral problem**: homogeneous wave equation with purely outgoing boundary conditions.
- Outgoing BCs naturally imposed at $\mathscr{I}^+$.
- Outgoing BCs actually "incorporated" at $\mathscr{I}^+$:
  - Geometrically: null cones outgoing.
  - Analytically: BCs encoded into a singular operator, **"BCs as regularity conditions"**.
- **Eigenfunctions** do not diverge when $x \to \infty$: actually **integrable**. Key to Hilbert space.



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Hyperboloidal approach to QNMs

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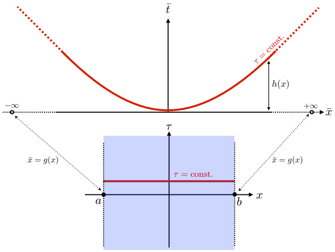

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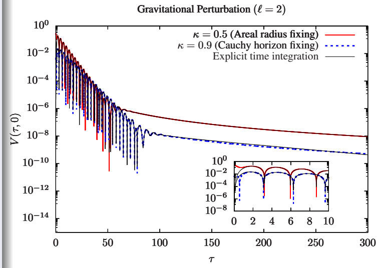

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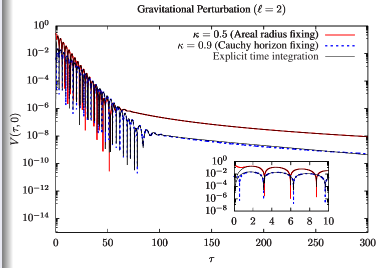

1. Wave problem: Pöschl-Teller example $V = V_o \mathrm{sech}^2(x)$

As starting point, consider the problem for a ϕ:

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + V\right)\phi = 0 \quad , \quad t \in]-\infty, \infty[\ , \ x \in]-\infty, \infty[$$

Hyperboloidal approach to QNMs: exploratory example

2. Choice of hyperboloidal foliation and compactification

Make the change to Bizoń-Mach variables [Bizoń & Mach 17]:

$$\begin{cases} \tau = t - \ln(\cosh x) \\ \bar{x} = \tanh x \end{cases} , \quad \tau \in]-\infty, \infty[\ , \ \bar{x} \in]-1, 1[$$

1. $\tau = \text{const.}$ defines a hyperboloidal slicing.
2. Compactification along hyperboloidal slices.

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3. We add the boundaries $\bar{x} = \pm 1$.

3. Wave equation in hyperboloidal coordinates: no boundary conditions allowed

For $\bar{x} = \pm 1$, $V = 0$. In the interior, $\bar{x} \in]-1, 1[$:

$$\left(\partial_\tau^2 + 2\bar{x}\partial_\tau\partial_{\bar{x}} + \partial_\tau + 2\bar{x}\partial_{\bar{x}} - (1-\bar{x}^2)\partial_{\bar{x}}^2 + \tilde{V}\right)\phi = 0 \ ,$$

with $\tilde{V} = \dfrac{V}{1-\bar{x}^2} = \dfrac{V_o \text{sech}^2(x)}{1-\bar{x}^2} = \dfrac{V_o(1-\bar{x}^2)}{1-\bar{x}^2} = V_o$.

Wave equation: reduction to first order system

4. Evolution equation in first order form

Introducing the auxiliary field

$$\psi = \partial_\tau \phi \ ,$$

we can write the wave equation in first-order form:

$$\partial_\tau \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \left(\begin{array}{c|c} 0 & 1 \\ \hline (1-\bar{x}^2)\partial_{\bar{x}}^2 - 2\bar{x}\partial_{\bar{x}} - V_o & -(2\bar{x}\partial_{\bar{x}} + 1) \end{array} \right) \begin{pmatrix} \phi \\ \psi \end{pmatrix} \ .$$

Spectral problem: first order formulation

5. Eigenvalue problem for a non-selfadjoint operator, no BCs

Our problem: study

$$L\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \omega \begin{pmatrix}\phi\\ \psi\end{pmatrix} \quad , \quad L = \left(\begin{array}{c|c} 0 & 1 \\ \hline L_1 & L_2 \end{array}\right) ,$$

where

$$L_1 = (1-\bar{x}^2)\partial^2_{\bar{x}} - 2\bar{x}\partial_{\bar{x}} - V_o \quad , \quad L_2 = -(2\bar{x}\partial_{\bar{x}} + 1) \ .$$

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Methodology here: "exploration" stage

Numerical spectral methods: **Chebyshev polynomials** truncations.

Calculate the spectrum of matrix $L^N \in M_N(\mathbb{R})$ approximants of L.

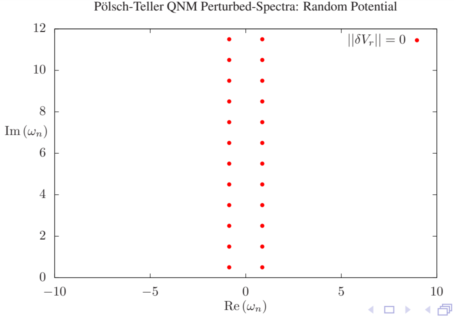
Test-bed study: Pöschl-Teller potential

Pöschl-Teller potential [JLJ, Macedo & Al Sheikh 21] (toy-model in [Bizoń, Chmaj & Mach 20])

$$V = V_o \, \mathrm{sech}^2(x) = V_o(1 - \bar{x}^2) \implies \boxed{\tilde{V} = V_o}$$

Particularly simple form (scalar field in de Sitter, $m^2 = V_o$ [Bizoń, Chmaj & Mach 20])

- Integrable potential (QNM completeness [Beyer 99]with $m^2 = V_o$!).
- QNM frequencies: $\omega_n^\pm = -is_n^\pm = \pm\frac{\sqrt{3}}{2} + i\left(n + \frac{1}{2}\right)$
- Here, eigenfunctions are Jacobi polynomials: $\phi_n(\bar{x}) = P_n^{(s_n^\pm, s_n^\pm)}(\bar{x})$.



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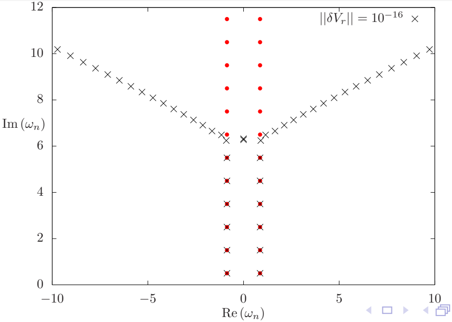

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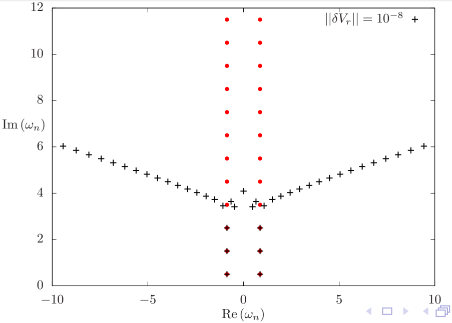

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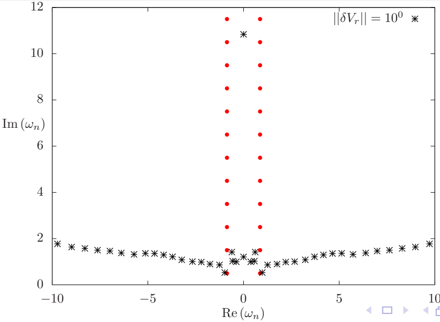

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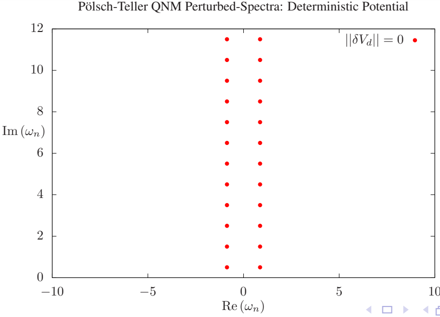

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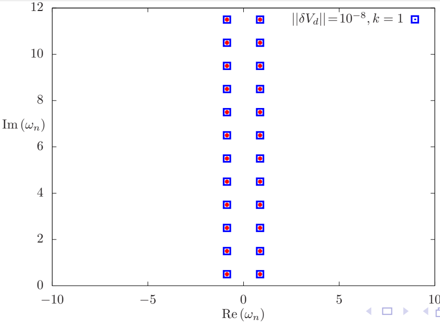

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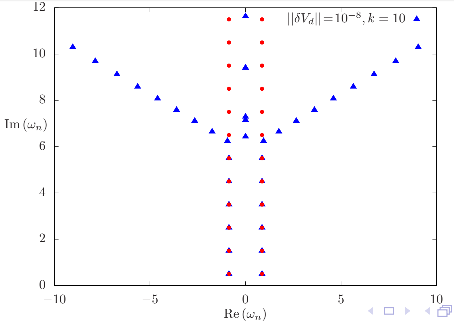

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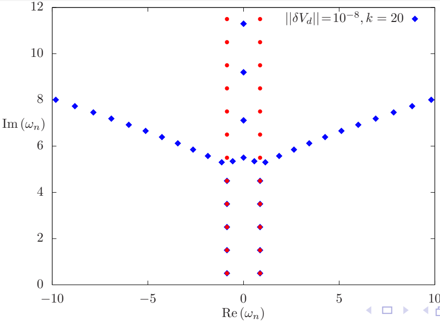


QNM: (spherically symm.) "general case" [JLJ, Macedo, Al Sheikh 21]

Starting point: (scalar) wave equation in "tortoise" coordinates

On a stationary spatime (with timelike Killing ∂_t):

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2} + V_\ell\right)\phi_{\ell m} = 0 \ ,$$

Dimensionless coordinates: $\bar{t} = t/\lambda$ and $\bar{x} = r_*/\lambda$ (and $\bar{V}_\ell = \lambda^2 V_\ell$),

Hyporboloidal approach

$$\left\{\begin{array}{rcl} \bar{t} & = & \tau - h(x) \\ \bar{x} & = & f(x) \end{array}\right. \quad .$$

- $h(x)$: implements the hyperboloidal slicing, i.e. $\tau = \mathrm{const.}$ is a horizon-penetrating hyperboloidal slice Σ_τ intersecting future $\mathscr{I}^+$.
- $f(x)$: spatial compactification between $\bar{x} \in [-\infty, \infty]$ to $[a, b]$.
- Timelike Killing: $\lambda\partial_t = \partial_{\bar{t}} = \partial_\tau$.

QNM: (spherically symm.) "general case" [JLJ, Macedo, Al Sheikh 21]

First-order reduction: $\psi_{\ell m} = \partial_\tau \phi_{\ell m}$

$$\boxed{\partial_\tau u_{\ell m} = iLu_{\ell m}} \quad , \quad \text{with} \quad u_{\ell m} = \begin{pmatrix} \phi_{\ell m} \\ \psi_{\ell m} \end{pmatrix}$$

where

$$\boxed{L = \frac{1}{i}\left(\begin{array}{c|c} 0 & 1 \\ \hline L_1 & L_2 \end{array}\right)}$$

$$\begin{aligned} L_1 &= \frac{1}{w(x)}\left(\partial_x\left(p(x)\partial_x\right) - q(x)\right) \qquad \text{(Sturm-Liouville operator)} \\ L_2 &= \frac{1}{w(x)}\left(2\gamma(x)\partial_x + \partial_x\gamma(x)\right) = \frac{1}{w(x)}\left(\gamma(x)\partial_x + \partial_x(\gamma(x)\cdot)\right) \end{aligned}$$

with $w(x) = \dfrac{f'^2 - h'^2}{|f'|} > 0$, $p(x) = \dfrac{1}{|f'|}$, $q(x) = |f'|\, V_\ell$, $\gamma(x) = \dfrac{h'}{|f'|}$.

QNM: (spherically symm.) "general case" [JLJ, Macedo, Al Sheikh 21]

Spectral problem

Taking Fourier transform, dropping (ℓ, m) (convention $u(\tau, x) \sim u(x)e^{i\omega\tau}$):

$$\boxed{L\, u_n = \omega_n\, u_n}\ .$$

where

$$\boxed{L = \frac{1}{i}\left(\begin{array}{c|c} 0 & 1 \\ \hline L_1 & L_2 \end{array}\right)}$$

$$L_1 = \frac{1}{w(x)}\left(\partial_x\left(p(x)\partial_x\right) - q(x)\right) \qquad \text{(Sturm-Liouville operator)}$$

$$L_2 = \frac{1}{w(x)}\left(2\gamma(x)\partial_x + \partial_x\gamma(x)\right)$$

Hyperboloidal approach: **No boundary conditions**

It holds $p(a) = p(b) = 0$, L_1 is "singular": **BCs "in-built" in** L.

QNM: (spherically symm.) "general case" [JLJ, Macedo, Al Sheikh 21]

A "natural" scalar product: initial value problem and physical content

Natural scalar product (where $\tilde{V}_\ell := q(x) > 0$):

$$\langle u_1, u_2 \rangle_E = \frac{1}{2}\int_a^b \Big(w(x)\bar{\psi}_1\psi_2 + p(x)\partial_x\bar{\phi}_1\partial_x\phi_2 + \tilde{V}_\ell\bar{\phi}_1\phi_2 \Big) dx \ ,$$

associted with the "total energy" of ϕ on Σ_t, defining the **"energy norm"**

$$||u||_E^2 = \langle u, u \rangle_E = \int_{\Sigma_\tau} T_{ab}(\phi, \partial_\tau\phi) t^a n^b d\Sigma_\tau \ ,$$

Spectral problem of a non-selfadjoint operator

- Full operator L: not selfadjoint.
- L_2: dissipative term encoding the energy leaking at $\mathscr{I}^+$.
- L selfadjoint in the non-dissipative $L_2 = 0$ case.

QNM: (spherically symm.) "general case" [JLJ, Macedo, Al Sheikh 21]

Adjoint operator

$$L^\dagger = \frac{1}{i}\left(\begin{array}{c|c} 0 & 1 \\ \hline L_1 & L_2 + L_2^\partial \end{array}\right)$$

where

$$L_2^\partial = 2\frac{\gamma}{w}\Big(\delta(x-a) - \delta(x-b)\Big)$$

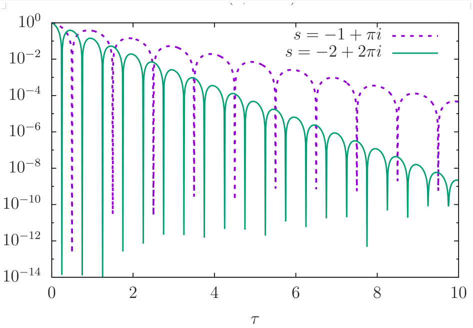
Loss of "self-adjointness" happens at the boundaries (as expected)

Scheme

Need of enhaced regularity

Schwarzschild BH QNMs: Ansorg & Macedo [Ansorg & Macedo 16]

- If only C^∞-regularity is required, then every point in the upper complex plane is an "eigenvalue".
- In particular, initial data can be chosen that decay in an exponentially damped sinusoid with arbitrary complex frequency.
- Need to require more regularity.



Enhanced regularity and choice of Hilbert space

Asymptotically AdS (and dS) BH spacetimes: Warnick [Warnick 15]

Discrete H^k-QNM spectrum: $Lu_n = \omega_n u_n$. Horizontal band structure of QNMs:

$$\mathrm{Im}(\omega_n) < \frac{1}{2} w_L + \kappa \left(k - \frac{1}{2}\right) \quad , \quad u_n \in H^k(\Sigma)$$

All ω above the band $\frac{1}{2} w_L + \kappa_n (k - \frac{1}{2})$ are valid eigenfrequencies at this H^k-regularity level. **Higher regularity required for higher QNM overtones**.

Asymptotically flat BH spacetimes: Warnick & Gajic [Warnick & Gajic 20] Galkowski & Zworski 21 [Galkowski & Zworski 21]

Identification of **Gevrey-2** regularity classes for QNMs in (**asymptotically flat**) hyperboloidal framework (complemented in [Galkowski & Zworski 21] in a complex scaling setting).

Hilbert spaces of $(\sigma, 2)$-Gevrey functions on $[0, \rho_0]$: f, $g \in C^\infty((0, \rho_0); \mathbb{C})$

$$\langle f, g \rangle_{G^2_{\sigma,1,\rho_o}} = \sum_{n=0}^{\infty} \frac{\sigma^{2n}}{n!^2 (n+1)!^2} \int_0^{\rho_0} \partial_\rho^n f \partial_\rho^n \bar{g} d\rho$$

The Anti-de Sitter (and de Sitter) case: H^k regularity

AdS-BH QNMs and regularity: κ-width horizontal bands

- The upper complex plane has a structure in horizontal bands of κ (surface gravity) width.
- QNMs contained in the first k bands are required to have a H^k regularity.
- As we go up (increase $\mathrm{Im}(\omega)$), more regularity is required pick up the QNMs ω_n's among all ω's.
- Above the first k bands, all ω's are H^k-QNMs.

The Anti-de Sitter (and de Sitter) case: some details

Consider the hyperbolic initial value problem on a hyperboloid:

$$\begin{cases} a\left(\Box_g \phi + V\phi\right) + W(\phi) = 0 \\ \phi(\tau = 0) = \phi_o \\ \psi(\tau = 0) = \psi_o \end{cases}$$

giving rise to the (semigroup) solution operator $\mathcal{S}(\tau)$

$$\begin{array}{rccc} \mathcal{S}(\tau): & H^1 \times H^0 & \to & H^1 \times H^0 \\ & (\phi_o, \psi_o) & \mapsto & (\phi(\tau), \partial_\tau \phi(\tau)) \end{array}$$

Definition (operator L)

$$D^k(L) := \left\{ (\phi, \psi) \in H^k \times H^{k-1} : \lim_{\tau \to 0^+} \frac{\mathcal{S}(\tau)\begin{pmatrix}\phi \\ \psi\end{pmatrix} - \begin{pmatrix}\phi \\ \psi\end{pmatrix}}{t} \text{ exists in } H^k \times H^{k-1} \right\}$$

$$L := \lim_{\tau \to 0^+} \frac{\mathcal{S}(\tau)\begin{pmatrix}\phi \\ \psi\end{pmatrix} - \begin{pmatrix}\phi \\ \psi\end{pmatrix}}{t} = \left(\begin{array}{c|c} 0 & 1 \\ \hline L_1 & L_2 \end{array}\right) : H^k \times H^{k-1} \to H^k \times H^{k-1}$$

The Anti-de Sitter (and de Sitter) case: some details

Definition H^k-QNM in AdS (or dS) BHs

Let $(D^k(L), L)$ be the infinitesimal generator of the solution semigroup $\mathcal{S}(\tau)$ on $H^k \times H^{k-1}$. Then $\omega \in \mathbb{C}$ is said to belong to the H^k-QNMs if:

(i) $\mathrm{Im}(\omega) < \dfrac{1}{2}w_L + \kappa\left(k - \dfrac{1}{2}\right)$

(ii) ω belongs to the spectrum of $(D^k(L), L)$.

Theorem [Warnick 15]

Let $(D^k(L), L)$ be the infinitesimal generator of the solution semigroup on $H^k \times H^{k-1}$, then:

(i) For $\mathrm{Im}(\omega) < \dfrac{1}{2}w_L + \kappa\left(k - \dfrac{1}{2}\right)$, κ de BH surface gravity, either:

- a) ω belongs to the resolvent set of $(D^k(L), L)$, (the **resolvent** $(L-\omega)^{-1}$ is a **bounded** linear transformation of $H^k \times H^{k-1}$ onto $D^k(L)$).
- b) ω is an eigenvalue of $(D^k(L), L)$ with finite multiplicity.

(ii) If $k_1 \geq k_2$, the eigenvalues and eigenfunctions of $(D^{k_1}(L), L)$ and $(D^{k_2}(L), L)$ agree for $\mathrm{Im}(\omega) < \frac{1}{2}w_L + \kappa\left(k - \frac{1}{2}\right)$.

Scheme

Spectral Theorem. Normal and 'non-normal' operators

Normal operators: Spectral Theorem

- **Normality**: denoting the adjoint matrix by $L^\dagger$, then L is normal iff

$$[L, L^\dagger] = LL^\dagger - L^\dagger L = 0$$

  Matrix examples: symmetric, hermitian, orthogonal, unitary...

- **Spectral Theorem** ("moral statement"):
  L is normal iff is unitarily diagonalisable.

Note: this depends on the adjoint $L^\dagger$, then on the Hilbert space (scalar product).

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'Non-normal' operators, $[L, L^\dagger] \neq 0$: no Spectral Theorem

- Completeness more difficult to study.
- Eigenvectors not necessarily orthogonal.
- **Spectral instabilities**.

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Methodology here: "exploration" stage

Numerical spectral methods: **Chebyshev polynomials** truncations.

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Example of spectral instability

$$L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c \quad , \quad a, b, c \in \mathbb{R}$$

acting on functions in $L^2([0,1])$, with homogeneous Dirichlet conditions (Chebyshev finite-dimensional matrix approximates).

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Example of spectral instability

$$L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c + \epsilon E_{\mathrm{Random}} \quad , \quad a, b, c \in \mathbb{R}, \ ||E_{\mathrm{Random}}|| = 1$$

acting on functions in $L^2([0,1])$, with homogeneous Dirichlet conditions (Chebyshev finite-dimensional matrix approximates).

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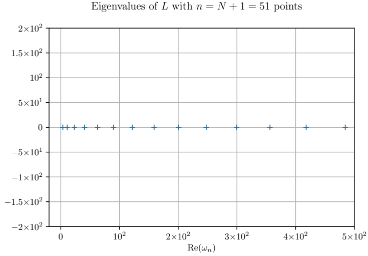

$$a = -1, \ b = 0, \ c = 1, \ \epsilon = 0$$

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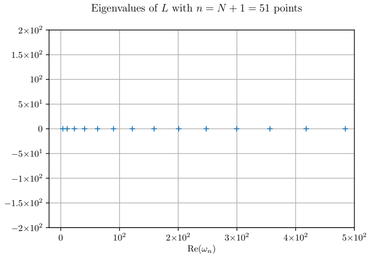

$$a = -1, \; b = 0, \; c = 1, \; \epsilon = 10^{-5}$$

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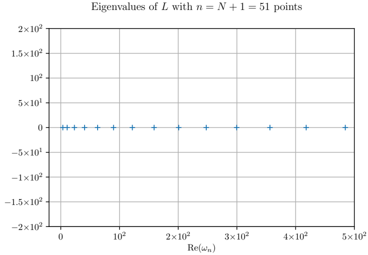

$$a = -1, \ b = 0, \ c = 1, \ \epsilon = 10^{-2}$$

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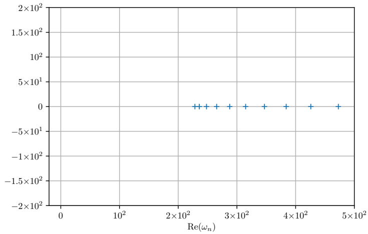
Eigenvalues of L with $n = N + 1 = 51$ points



$$a = -1, \ b = 30, \ c = 1, \ \epsilon = 0$$

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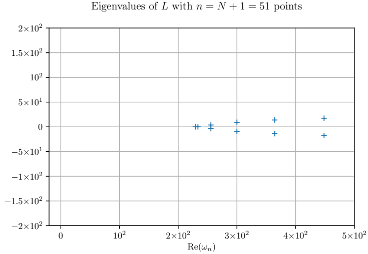

$$a = -1, \; b = 30, \; c = 1, \; \epsilon = 10^{-10}$$

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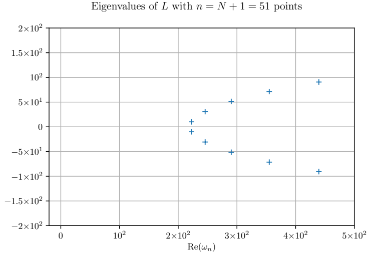

$$a = -1, \; b = 30, \; c = 1, \; \epsilon = 10^{-8}$$

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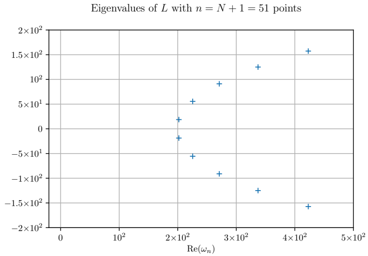

$$a = -1, \; b = 30, \; c = 1, \; \epsilon = 10^{-6}$$

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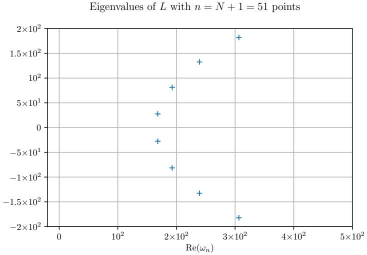

$$a = -1, \ b = 30, \ c = 1, \ \epsilon = 10^{-4}$$

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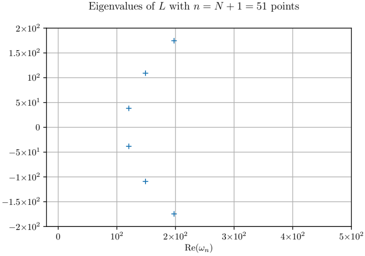

$$a = -1,\ b = 30,\ c = 1,\ \epsilon = 10^{-2}$$

Spectral (in)stability: eigenvalue condition number

Left- and right-eigenvectors, respectively u_i and v_i, of A

$$A^\dagger u_i = \bar{\lambda}_i u_i \ \ (\Leftrightarrow u_i^\dagger A = \lambda_i u_i^\dagger) \ \ , \ \ A v_i = \lambda_i v_i \ \ , \ \ i \in \{1, \ldots, n\} \ ,$$

Perturbation theory of eigenvalues [cf. Kato 80, ...; e.g. Trefethen, Embree 05]:

$$A(\epsilon) \ = \ A + \epsilon \delta A \ \ , \qquad ||\delta A|| = 1 \ .$$

$$|\lambda_i(\epsilon) - \lambda_i| = \epsilon \frac{|\langle u_i, \delta A \ v_i(\epsilon)\rangle|}{|\langle u_i, v_i\rangle|} \leq \epsilon \frac{||u_i|| \ ||\delta A \ v_i||}{|\langle u_i, v_i\rangle|} + O(\epsilon^2) \leq \epsilon \frac{||u_i|| \ ||v_i||}{|\langle u_i, v_i\rangle|} + O(\epsilon^2).$$

Eigenvalue condition number: $\kappa(\lambda_i)$

$$\boxed{\kappa(\lambda_i) = \frac{||u_i|| ||v_i||}{|\langle u_i, v_i\rangle|}}$$

Spectral (in)stability and Pseudospectrum

Pseudospectrum

Given $\epsilon > 0$, the ϵ-pseudospectrum $\sigma_\epsilon(L)$ of L is defined as [e.g Trefethen & Embree 05]:

$$\begin{aligned}\sigma_\epsilon(L) &= \boxed{\{\lambda \in \mathbb{C}, \text{ such that } \lambda \in \sigma(L + \delta L) \text{ for some } \delta L \text{ with } ||\delta L|| < \epsilon\}} \\ &= \{\lambda \in \mathbb{C}, \text{ such that } ||Lv - \lambda v|| < \epsilon \text{ for some } v \text{ with } ||v|| = 1\} \\ &= \{\lambda \in \mathbb{C}, \text{such that } ||(\lambda I - L)^{-1}|| > \epsilon^{-1}\}\end{aligned}$$

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Normal case: bounds on the norm of the resolvent $R_L(\lambda) = (\lambda I - L)^{-1}$

Given $\lambda \in \mathbb{C}$ and $\sigma(L)$ the spectrum of L, it holds

$$||(\lambda I - L)^{-1}||_2 = \frac{1}{\text{dist}(\lambda, \sigma(L))}$$

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&= \boxed{\{\lambda \in \mathbb{C}, \text{such that } ||(\lambda I - L)^{-1}|| > \epsilon^{-1}\}}
\end{aligned}$$

Non-normal case: bad control on the resolvent $R_L(\lambda)$. **Pseudospectrum**

The norm of the resolvent can become very large far from the spectrum:

$$||(\lambda I - L)^{-1}||_2 \leq \frac{\kappa}{\mathrm{dist}(\lambda, \sigma(L))}$$

where κ is a "condition number" assessing the lack of proportionality of 'left' and 'right' eigenvectors of L, and can become very large in the non-normal case.

Bauer-Fike theorem. Random perturbations

Pseudospectrum and condition number: Bauer-Fike theorem [e.g Trefethen & Embree 05]

Defining "tubular neighbourhood" of radius ϵ around $\sigma(A)$

$$\Delta_\epsilon(A) = \{\lambda \in \mathbb{C} : \mathrm{dist}\,(\lambda, \sigma(A)) < \epsilon\} \ ,$$

it holds: $\Delta_\epsilon(A) \subseteq \sigma_\epsilon(A)$. For normal operators: $\sigma_\epsilon(A) = \Delta_\epsilon(A)$.
Non-normal case, $\kappa(\lambda_i) \neq 1$, it holds (for small ϵ):

$$\sigma_\epsilon(A) \subseteq \bigcup_{\lambda_i \in \sigma(A)} \Delta_{\epsilon\kappa(\lambda_i)+O(\epsilon^2)}(\{\lambda_i\}) \ ,$$

Therefore $\sigma_\epsilon(A)$ larger tubular neighbourhood of radius $\sim \epsilon\kappa(\lambda_i)$.

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Random perturbations and Pseudospectrum

Random perturbations ΔL with $||\delta L|| < \epsilon$ "push" eigenvalues into $\sigma_\epsilon(A)$, providing an insightful and systematic manner of exploring $\sigma_\epsilon(L)$.

Bauer-Fike theorem. Random perturbations

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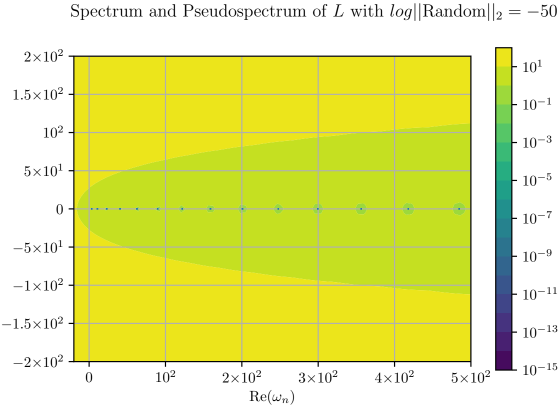
The 'role' of random perturbations [Sjöstrand 19; Hager 05, Montrieux, Nonnenmacher, Vogel,...]

Random perturbations improve the analytical behaviour of $R_L(\lambda)$!!!

Spectral (in)stability and Pseudospectrum: illustration

Pseudospectrum of: $L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c + \epsilon E_{\mathrm{Random}}$



$a = -1, \ b = 0, \ c = 1, \ \epsilon = 0$

Spectral (in)stability and Pseudospectrum: illustration

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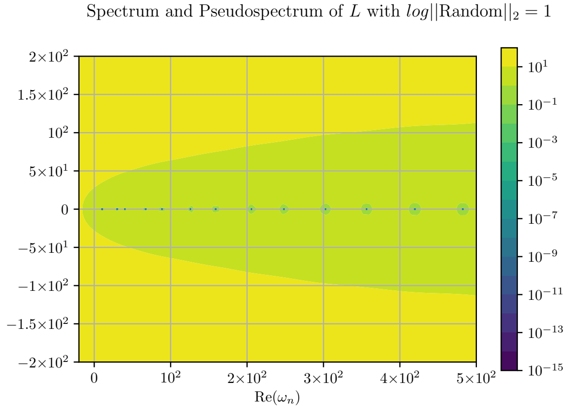
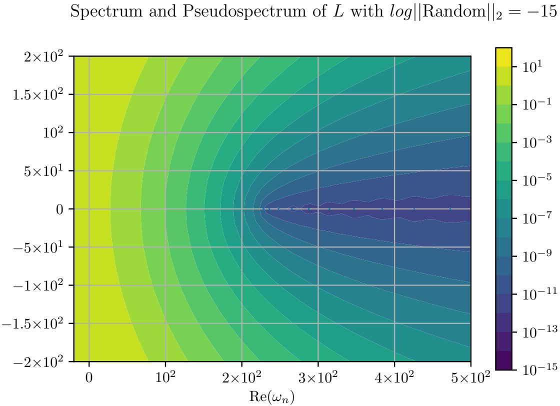


$a = -1, \; b = 0, \; c = 1, \; \epsilon = 10^{1}$

Spectral (in)stability and Pseudospectrum: illustration

Pseudospectrum of: $L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c + \epsilon E_{\mathrm{Random}}$

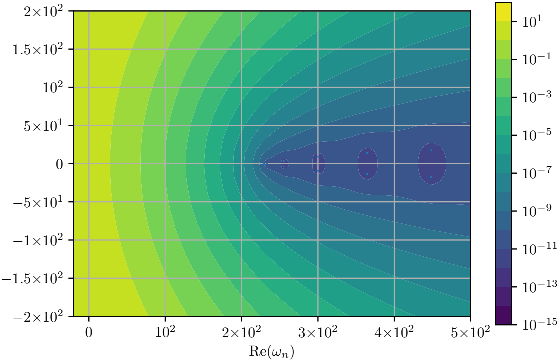


$a = -1, \; b = 30, \; c = 1, \; \epsilon = 0$

Spectral (in)stability and Pseudospectrum: illustration

Pseudospectrum of: $L = a\dfrac{d^2}{dx^2} + b\dfrac{d}{dx} + c + \epsilon E_{\mathrm{Random}}$

Spectrum and Pseudospectrum of L with $log||\mathrm{Random}||_2 = -10$

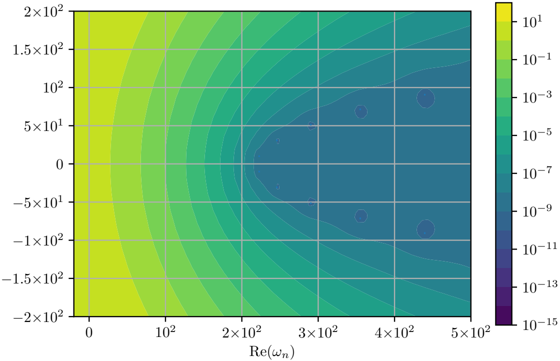


$a = -1,\ b = 30,\ c = 1,\ \epsilon = 10^{-10}$

Spectral (in)stability and Pseudospectrum: illustration

Pseudospectrum of: $L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c + \epsilon E_{\mathrm{Random}}$

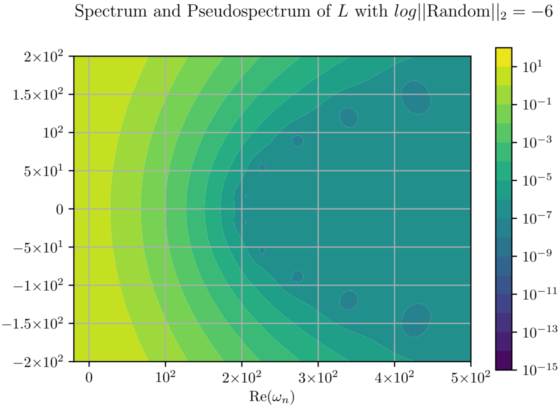
Spectrum and Pseudospectrum of L with $log||\mathrm{Random}||_2 = -8$



$a = -1, \; b = 30, \; c = 1, \; \epsilon = 10^{-8}$

Spectral (in)stability and Pseudospectrum: illustration

Pseudospectrum of: $L = a\dfrac{d^2}{dx^2} + b\dfrac{d}{dx} + c + \epsilon E_{\mathrm{Random}}$

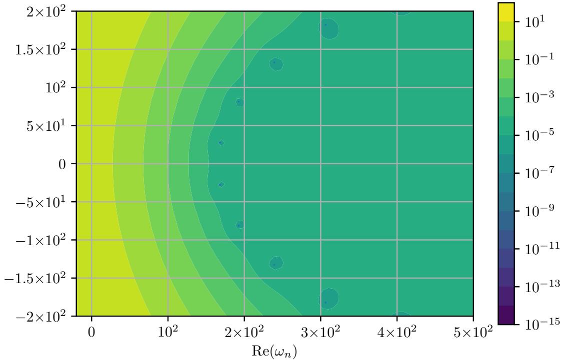


$a = -1, \; b = 30, \; c = 1, \; \epsilon = 10^{-6}$

Spectral (in)stability and Pseudospectrum: illustration

Pseudospectrum of: $L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c + \epsilon E_{\text{Random}}$

Spectrum and Pseudospectrum of L with $log||\text{Random}||_2 = -4$

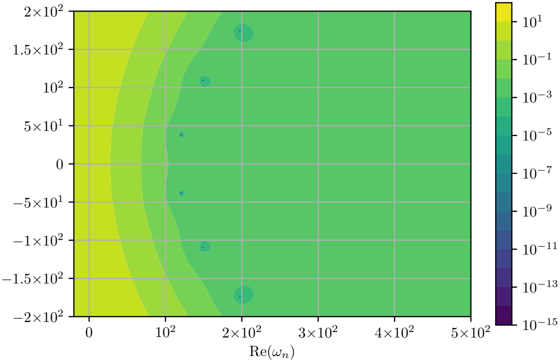


$a = -1,\ b = 30,\ c = 1,\ \epsilon = 10^{-4}$

Spectral (in)stability and Pseudospectrum: illustration

Pseudospectrum of: $L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c + \epsilon E_{\text{Random}}$

Spectrum and Pseudospectrum of L with $log||\text{Random}||_2 = -2$



$a = -1, \; b = 30, \; c = 1, \; \epsilon = 10^{-2}$

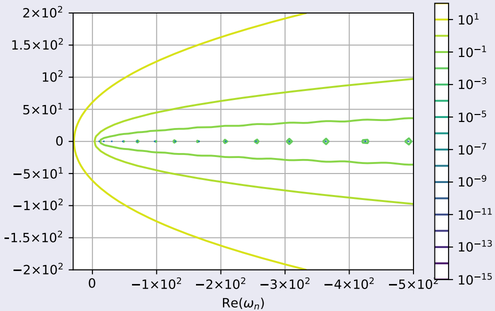
The relevance of the scalar product: assessing large/small

The illustrative operator: $L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c \quad , \quad a, b, c \in \mathbb{R}$

- Non-selfadjoint in standard $L^2([0,1])$ for $b \neq 0$.
- Formally normal!
- Non-normal: domain of $L^\dagger L$ and $LL^\dagger$ different.
- But actually self-adjoint...

Cast in Sturm-Liouville form: selfadjoint for appropriate scalar product $\langle \cdot, \cdot \rangle_w$!!!

Pseudospectrum using the L^2-inner-product



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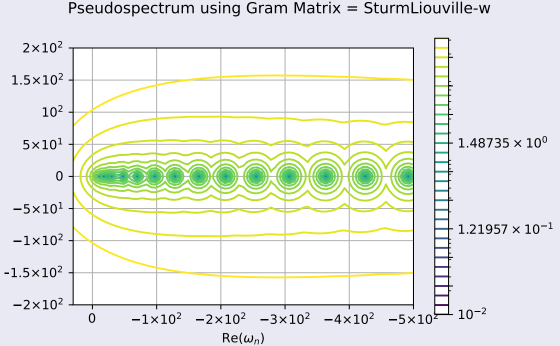
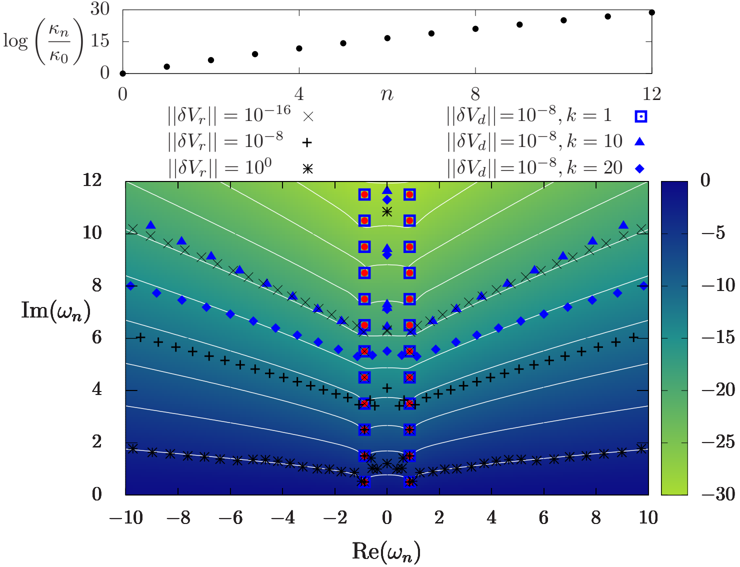
Cast in Sturm-Liouville form: selfadjoint in appropriate scalar product $\langle \cdot, \cdot \rangle_w$



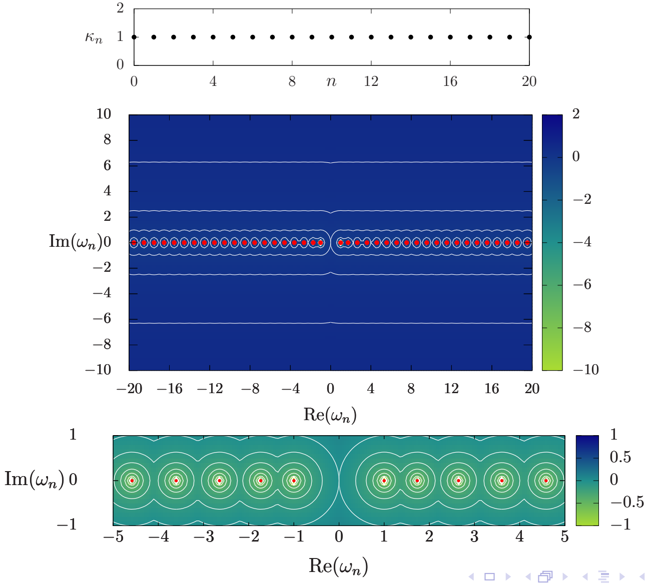
Pseudospectrum and QNM instability of BH spacetimes.

Pöschl-Teller, $L_2 \neq 0$:



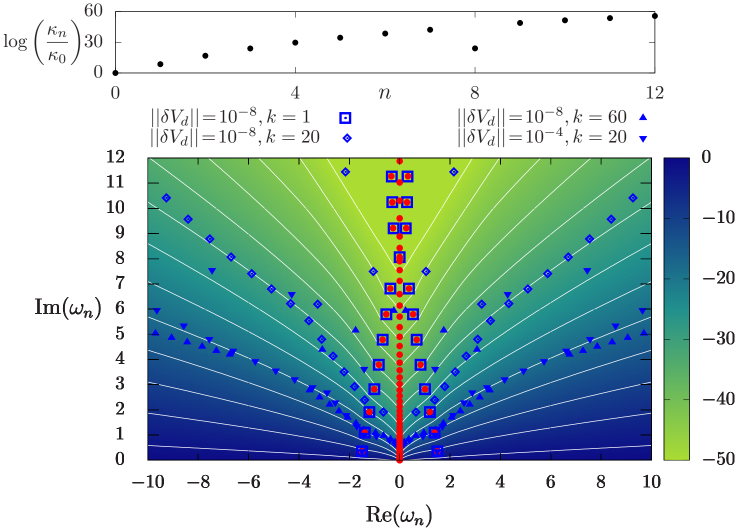
Pseudospectrum and QNM instability of BH spacetimes.

Pöschl-Teller, $L_2 = 0$:



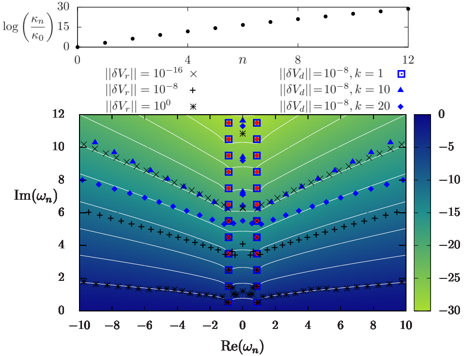
Pseudospectrum and QNM instability of BH spacetimes.

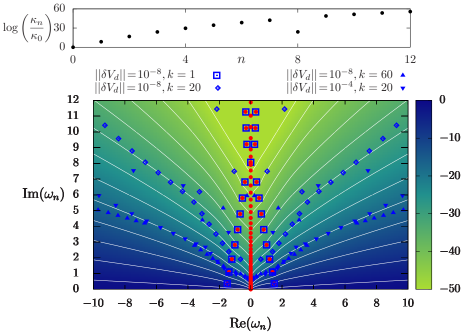
Schwarzschild ($s = 2$, $\ell = 2$):



Pseudospectrum and QNM instability of BH spacetimes.

Comparison Pöschl-Teller versus Schwarzschild:





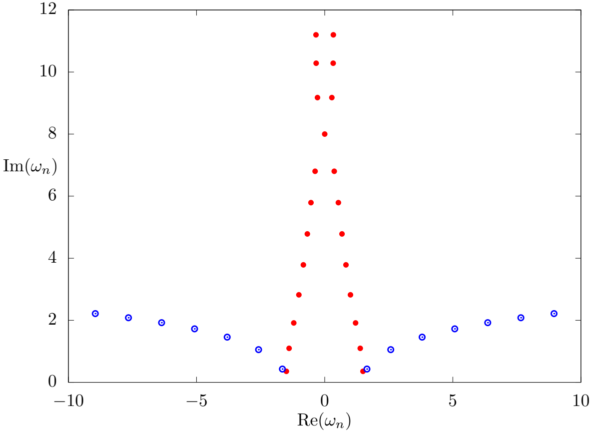
Remarks:

- High frequency perturbations: random δV_r, deterministic $\delta V_d \sim \cos(2\pi k x)$.
- **Fundamental QNM: ultraviolet stable, infrared unstable.**
- **QNM overtones: ultraviolet unstable.**
- **Migrate to Pseudospectrum contour lines: 'Universality' phenomenon?**

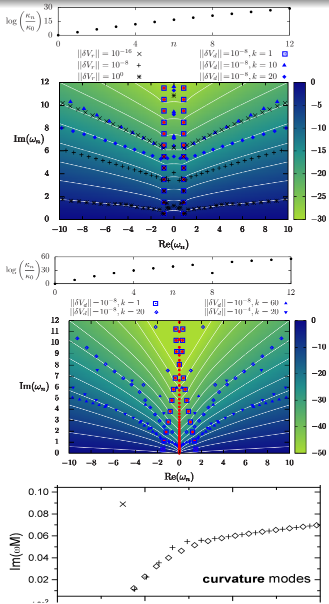
Pseudospectrum and QNM instability of BH spacetimes.

Infrared effect on fundamental QNM: Schwarzschild case

Comparison between QNMs for Schwarzschild potential (red circles) and QNMs of "cut-Schwarzschild", set to zero at large distances (blue circles).



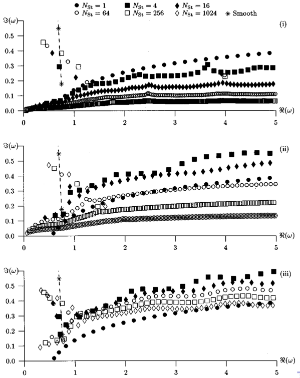
Pseudospectrum and QNM instability of BH spacetimes.



Black Hole and Neutron Star QNMs

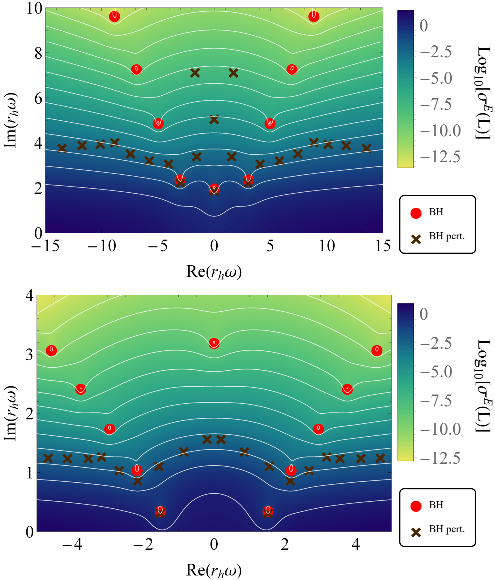
Comparison with:

- Nollert's high-frequency Schwarzschild perturbations.
- Nollert's remark on Neutron Stars (w-modes) curvature QNMs.



Pseudospectrum and QNM instability of BH spacetimes.

Schwarzschild - Anti-De Sitter (axial). No "branch cut"



- $\alpha = 1$
- Hyperboloidal slicing
- Energy norm

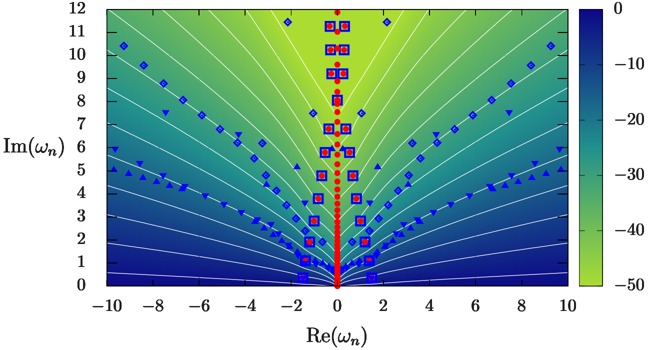
- $\alpha = 12$
- Hyperboloidal slicing
- Energy norm

QNMs and norms: "Definition" versus "Stability" problems

Stability of QNMs: we cannot choose the norm, the "physics" chooses for us

The system chooses what is a "big and small" perturbation: **here, the energy**.

$$||(\phi, \psi)||_E^2 = E = \int_\Sigma T_{ab}(\phi, \psi) t^a n^b d\Sigma \sim ||\phi||_{H^1_V} + ||\psi||_{L^2}$$



High-sensitivity of QNM overtones to low regularity background changes

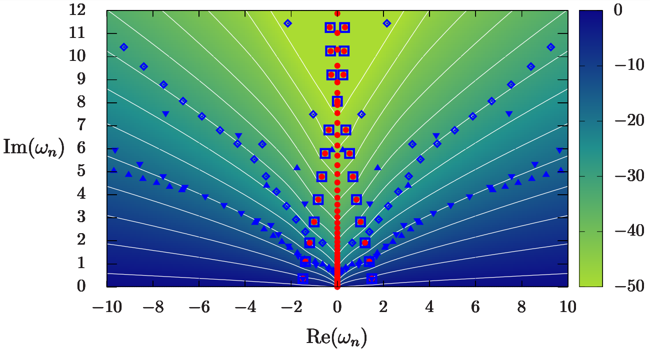
Numerical evidence: convergence of QNM overtones to perturbed frequencies.

QNMs and norms: "Definition" versus "Stability" problems

"Hints" for a reconciliation: the "Gevrey Ocean"

A small high-frequency perturbation in the energy, is a huge one in Gevrey terms.

- For a given V, **QNM eigenfunctions are defined to be in class Gevrey-2**.
- **Small (in the energy norm) perturbations** δV **of** V **lead to very different QNM** ω_n**'s**, respective eigenfunctions being indeed Gevrey-2, again.



"Energy-Pseudospectrum = Chart of the 'Gevrey Ocean' "

Scheme

Main Problem: convergence of the Pseudospectrum

A Major Issue

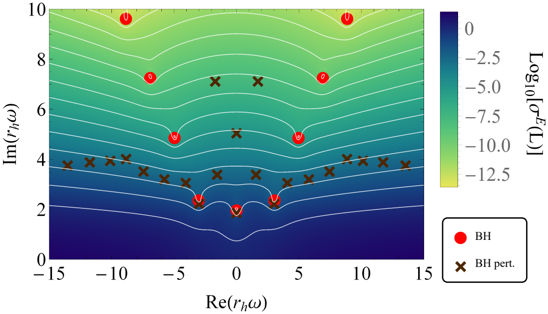
The "energy-Pseudospectrum" does not converge

.

That is, fixing ω in the upper half complex plane, the norm of the N-resolution numerical approximant to the resolvent $||R_L(\omega)||_{_E} = ||(L - \omega\,\mathrm{Id})^{-1}||_{_E}$ does not converge when $N \to \infty$.

- Either $R_L(\omega) = (L - \omega)^{-1}$ is not a compact operator, so we cannot approximate it by matrices
(in particular, this would happen in the spectrum $\sigma(L)$ is not dicrete)
- Or there is some subtlety in the limiting process, not addressed in our exploratory attempt (e.g. "limits of limits" in algorithms discussed by Ben-Artzi [cf. Ben-Artzi's talk on Monday]

Schwarzschild-Anti-de Sitter: hyperboloidal slicing

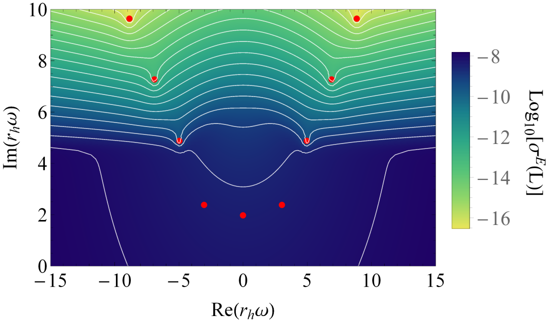


- $\alpha = 1, \kappa = 2$
- Hyperboloidal slicing
- Energy norm

Schwarzschild-AdS pseudospectra: hyperboloidal slicing

- Schwarzschild-AdS metric: $ds^2 = -f(r)dt^2 + f^{-1}(r) + r^2 d\Omega^2$, $\alpha = \dfrac{r_h}{R}$
  with $f(r) = 1 - \dfrac{2M}{r} + \dfrac{r^2}{R^2}$, $\alpha = \frac{r_h}{R}$, and $V_\ell(r) = \dfrac{f(r)}{r^2}\left(\ell(\ell+1) - \dfrac{6M}{r}\right)$.
- Spectral problem: $Lu = \omega u$.
- Pseudospectrum: contour map of $1/||(L-\omega)^{-1}||$

Schwarzschild-Anti-de Sitter: hyperboloidal slicing



- $\alpha = 1, \kappa = 2$
- Hyperboloidal slicing
- H^k, with $k = 4$.

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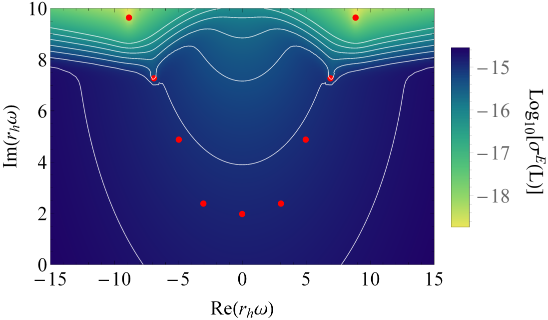
Schwarzschild-Anti-de Sitter: hyperboloidal slicing



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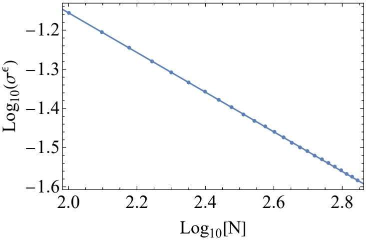
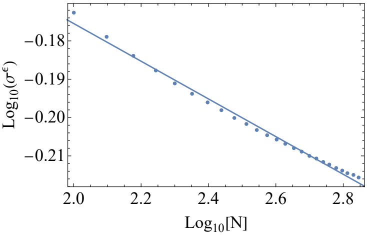
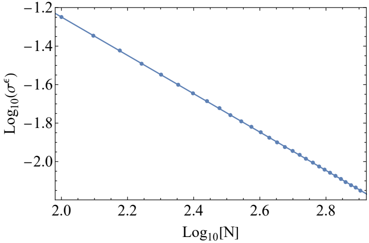


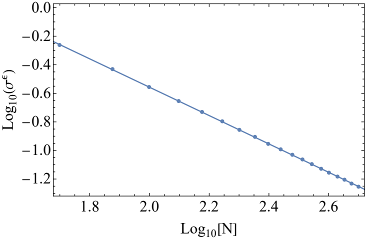
$H^k, k = 1; \omega = 2 + i0.5$



$H^k, k = 1; \omega = 2 - i0.5$

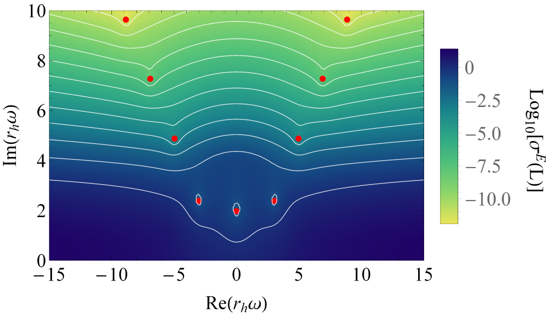


$H^k, k = 2; \omega = 2 - i0.5$



$H^k, k = 2; \omega = 2 - i6$

Schwarzschild-Anti-de Sitter: hyperboloidal slicing



- $\alpha = 1, \kappa = 2$
- Null slicing
- H^k, with $k = 2$.

Schwarzschild-AdS pseudospectra: null slicing

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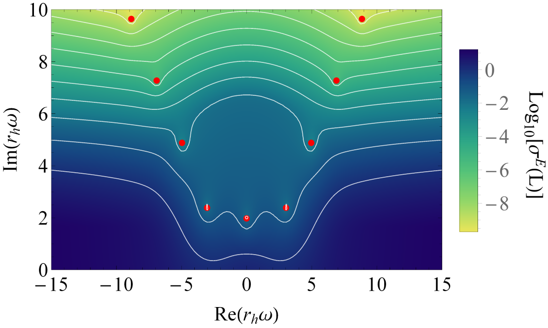
- Spectral problem: $Au = \omega Bu$.
- Pseudospectrum: contour map of $1/||(A - \omega B)^{-1}||$

Schwarzschild-Anti-de Sitter: hyperboloidal slicing

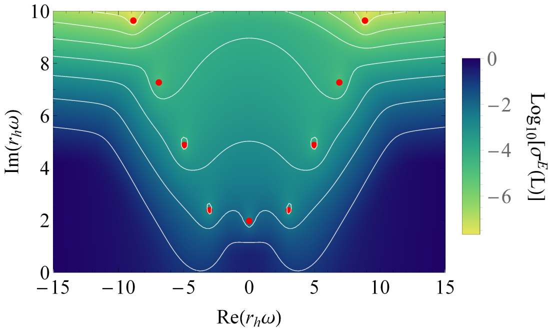


- $\alpha = 1, \kappa = 2$
- Null slicing
- H^k, with $k = 3$.

Schwarzschild-AdS pseudospectra: null slicing

- Schwarzschild-AdS metric: $ds^2 = -f(r)dt^2 + f^{-1}(r) + r^2 d\Omega^2$, $\alpha = \dfrac{r_h}{R}$ with $f(r) = 1 - \dfrac{2M}{r} + \dfrac{r^2}{R^2}$, $\alpha = \frac{r_h}{R}$, and $V_\ell(r) = \dfrac{f(r)}{r^2}\left(\ell(\ell+1) - \dfrac{6M}{r}\right)$.
- Spectral problem: $Au = \omega B u$.
- Pseudospectrum: contour map of $1/||(A - \omega B)^{-1}||$

Schwarzschild-Anti-de Sitter: hyperboloidal slicing



- $\alpha = 1, \kappa = 2$
- Null slicing
- H^k, with $k = 4$.

Schwarzschild-AdS pseudospectra: null slicing

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- Spectral problem: $Au = \omega Bu$.
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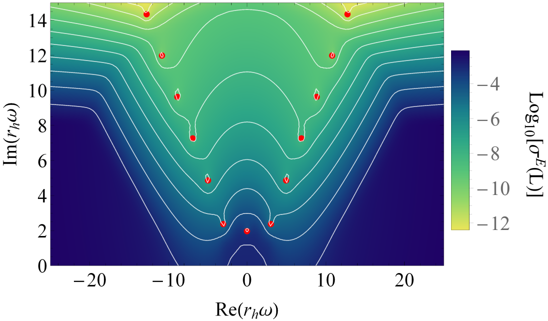
Schwarzschild-Anti-de Sitter: hyperboloidal slicing



- $\alpha = 1, \kappa = 2$
- Null slicing
- H^k, with $k = 6$.

Schwarzschild-AdS pseudospectra: null slicing

- Schwarzschild-AdS metric: $ds^2 = -f(r)dt^2 + f^{-1}(r) + r^2 d\Omega^2$, $\alpha = \dfrac{r_h}{R}$
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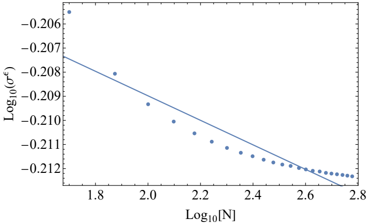
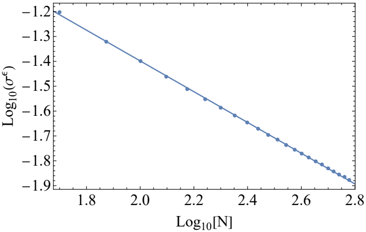
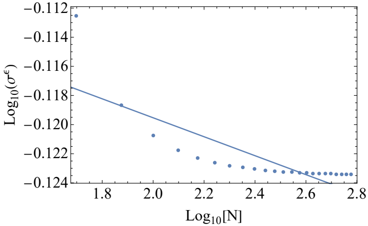
Schwarzschild-Anti-de Sitter: hyperboloidal slicing



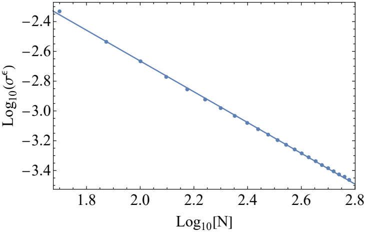
$H^k, k = 1; \omega = 2 + i1.5$



$H^k, k = 1; \omega = 2 + i2.6$

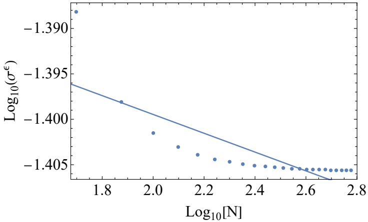


$H^k, k = 2; \omega = 8 + i3$

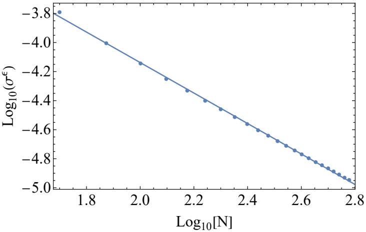


$H^k, k = 2; \omega = 8 + i5$

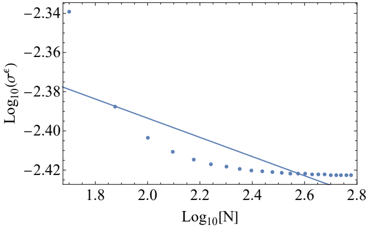
Schwarzschild-Anti-de Sitter: hyperboloidal slicing



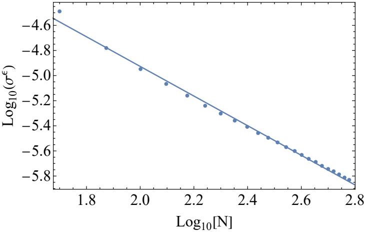
$H^k, k = 3; \omega = 8 + i5$



$H^k, k = 3; \omega = 8 + i7$

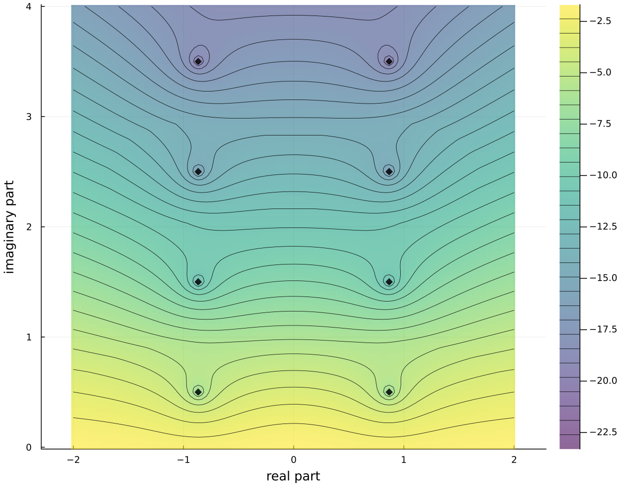


$H^k, k = 4; \omega = 14 + i7$



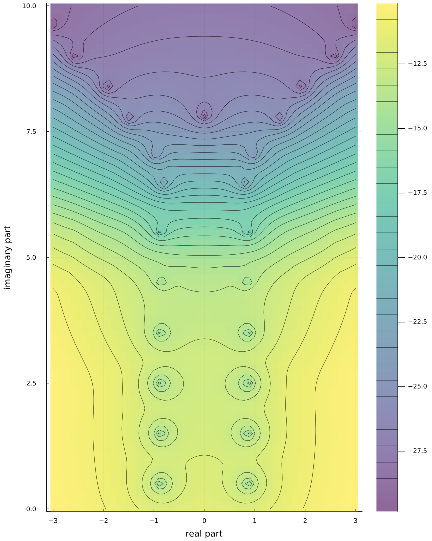
$H^k, k = 2; \omega = 14 + i9$

Pöschl-Teller and H^k-pseudospectra



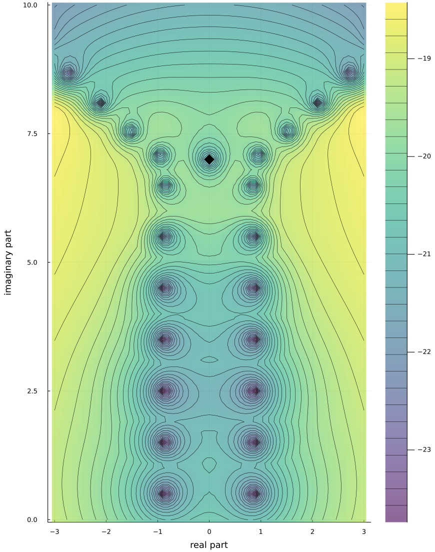
$H^k, k = 0$

Pöschl-Teller and H^k-pseudospectra



$H^k, k = 5$

Pöschl-Teller and H^k-pseudospectra



$H^k, k = 7$

Scheme

Pseudospectrum and QNM instability: Caveats

Vishveshwara, C.V., "On the black hole trail...: a personal journey"

18th Conference of the Indian Association for General Relativity and Gravitation, pp. 11-22 (1996)

wave during the coalscence of binary black holes[18]. Recently Aguirregabiria and I have studied the sensitivity of the quasinormal modes to the scattering potential[19]. The motivation is to understand how any perturbing influence, such as another gravitating source, that might alter the effective potential would thereby affect the quasinormal modes. Interestingly, we find that the fundamental mode is, in general, insensitive to small changes in the potential, whereas the higher modes could alter drastically. The fundamental mode would therefore carry the imprint of the black hole, while higher modes might indicate the nature of the perturbing source.

Quasinormal modes are perhaps the rebuttal to the criticism of my thesis examiner regarding the nonobservability of black holes.

Pseudospectrum and QNM instability: Caveats

Need of conceptual assessments

- **Non-convergence of the "energy-norm Pseudospectrum", whereas convergence of QNM "ultraviolet" perturbations**: need of assessment.

- **Is it realistic a stationary perturbation?** Time-fluctuating perturbation:
  $\partial_\tau u^\epsilon(\tau,x) = i\big(L(x) + \epsilon\delta L(\tau,x)\big)u^\epsilon(\tau,x)$
  $\partial_\tau \bar{u}^\epsilon(\tau,x) = i\big(L(x) + \epsilon\overline{\delta L}(x)\big)\bar{u}^\epsilon(\tau,x) \quad , \quad \overline{\delta L}(x) = \frac{1}{T}\int_0^T \delta L(t,x)d\tau$

- **GR perturbation theory**: Eqs. hierarchy on the same stationary background
  $$(\partial_\tau - iL)\,u^{(1)} = 0 \quad , \quad (\partial_\tau - iL)\,u^{(2)} = S(\tau,x;u^{(1)}),\ldots$$

- **Data Analysis challenge**:
  Detection and parameter estimation issues under the large parameter spaces involved and the universality properties of small scale perturbations.

Bottomline: need of other tools

Need to formalise the effect of low regularity (ultraviolet) effects in QNMs.

Conclusions and Perspectives

Conclusions

- Methodological need of separating the **"definition" and the "stability" problems** in QNMs.

- Numerical evidence of **instability of QNM overtones** under high-frequency perturbations in the effective potential: **Nollert-Price BH QNM branches**.

  Fundamental QNM ultraviolet stable. Infrared unstable.

- Need of assessing the **Pseudospectrum**: potential quantitative tool to probe spacetime perturbations from scattering data.
  (**Inverse scattering spirit**: size/frequency of the perturbation from the reading of the QNMs displacement from non-perturbed values.)

- Hints towards **Universality of compact-object QNM branches** in the infinite high-frequency limit: a low regularity problem.

- **Strong modifications to BH QNMs in a purely classical GR setting.**

Conclusions and Perspectives

Perspectives: "Ultraviolet & Infrared" Conjectures

- **"Ultraviolet": Regge QNM branches conjecture.**

  Generic small-scale perturbations push QNM towards universal Regge branches, reaching them in the infinite "perturbation frequency" limit.

$$\mathrm{Re}(\omega_n) \sim \pm\left(\frac{\pi}{L}n + \frac{\pi\gamma_p}{2L}\right)$$

$$\mathrm{Im}(\omega_n) \sim \frac{1}{L}\left[\gamma \ln\left(\left(\frac{\pi}{L}n + \frac{\pi\gamma_p}{2L}\right) + \frac{\pi\gamma_\Delta}{2L}\right) - \ln S\right]$$

Can we measure the regularity of spacetime?

- **"Infrared": QNM Pseudospectrum encoded in asymptotic symmetries**

  "Missing" degrees of freedom are accounted for by asymptotic symmetry charges at $\mathscr{I}^+$ (BMS, ...?) and the BH horizon (extended-BMS [Ashtekar, Lewandowski et al.. 22]). The universal features of QNM branches can be understood in terms of the phase space of these dynamical symmetries.

- **"Infrared-ultraviolet": BMS-charges as inverse scattering data**

  Asymptotic (BMS?) charges provide necessary scattering data for Inverse-Scattering reconstruction, in particular the (low-regularity) small-scale structure of the spacetime [cf. Petropoulos, Mason, Taujanskas, Geiler, Taujanskas; Damour, Barnich].