On Foldy-Lax models for time-domain scattering by multiple small particles

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(POEMS (INRIA-CNRS-ENSTA), INRIA, France)

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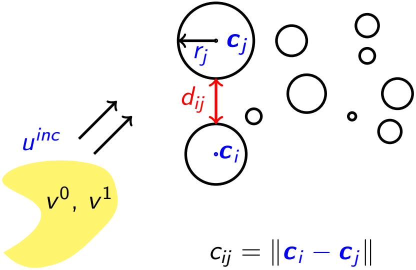
Please feel free to ask questions during the talk!

Problem Setting



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Particles are separated: $d_{ij} > 0 \ \forall i, j$.

Incident field

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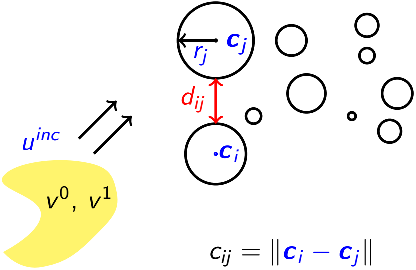

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Sound-soft scattering: find u

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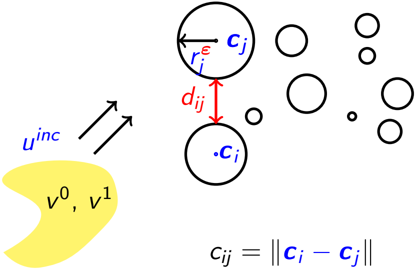

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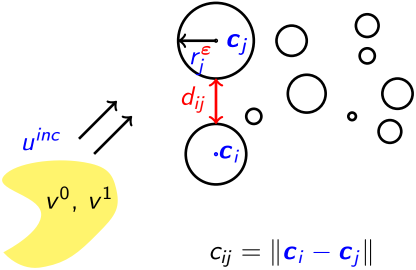

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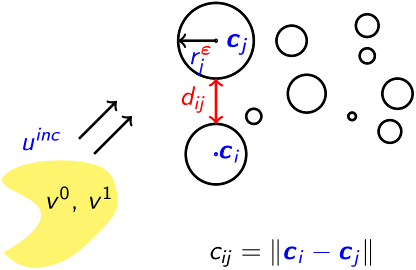

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Approximate $u = u^\varepsilon$ for $\varepsilon \to 0$ ($u^\varepsilon = O(\log^{-1} \varepsilon)$). Goal: $\|u_{app}^\varepsilon - u^\varepsilon\| / \|u^\varepsilon\| \to 0$.

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Such models have a potential in computations (cf. Barucq et al. 2021):

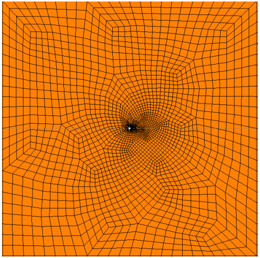

Figure: A tetrahedral mesh generated by GMSH in a domain with a small inclusion

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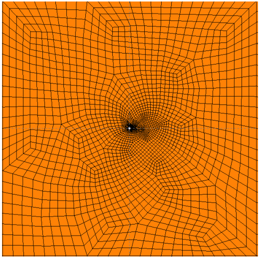

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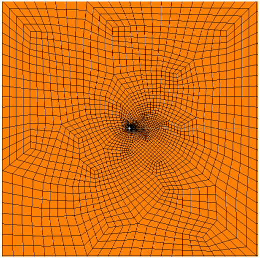

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- typical for wave propagation: **explicit** methods
  (for well-chosen spatial discretization mass matrix is easy to invert)
- CFL restriction: $\Delta t \lesssim Ch$
- small particles $\implies$ need to mesh finely in their vicinity+ CFL $\implies$ small time step
  (remedies: numerical (local time stepping / locally implicit methods) or analytical (asymptotic methods))

(Non-exhaustive) bibliography

Frequency-domain (Helmholtz with the frequency ω) (see also P. Martin, Multiple scattering. Interaction of time-harmonic waves with N obstacles):

1. generalized framework of Foldy, Lax (Foldy '45, Lax '51, '52), Cassier, Hazard '14
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This work: (mostly) 2D time-domain. We will derive a model as a Foldy-Lax model (explanation to follow).

- first part: circles
- second part (in progress): extension to particles of general shapes

Foldy-Lax in frequency domain: brief intro

Fourier-Laplace transform: $\mathcal{F}f(\omega) = \int_0^\infty e^{i\omega t} f(t)dt$, $\operatorname{Im}\omega \geq 0$.

2D Helmholtz $(-\Delta - \omega^2)$ fundamental solution: $G_\omega(r) := \frac{i}{4} H_0^{(1)}(\omega r)$

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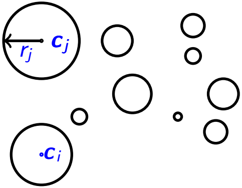

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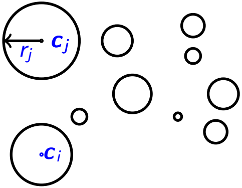

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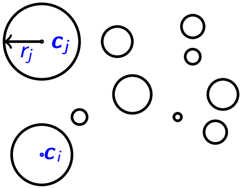

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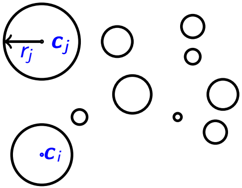

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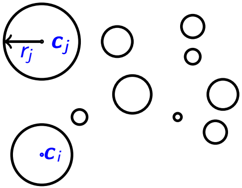

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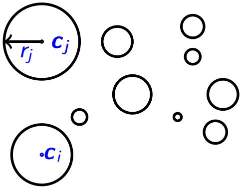

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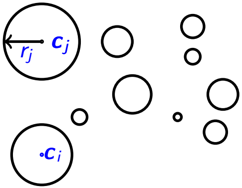

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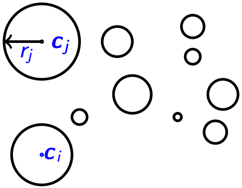

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Theorem (Cassier, Hazard '14, where the model is also derived rigorously)

$$-\hat{u}^{inc}(\boldsymbol{c}_k) = \hat{\lambda}^\varepsilon_k + \sum_{n\neq k} \hat{\lambda}^\varepsilon_n \frac{G_\omega(\|\boldsymbol{c}_k - \boldsymbol{c}_n\|)}{G_\omega(r_n)}, \quad k = 1, \dots, N.$$

Then, as $\varepsilon \to 0$, $\|\hat{u}^\varepsilon - \hat{u}^\varepsilon_{app}\|_{L^2(K)} \lesssim C_{\omega,K} \frac{\varepsilon}{|\log \varepsilon|}$ for all compact $K \subset \Omega^c$.

Foldy-Lax model in time domain: first idea

Frequency domain

$$\hat{u}_{FL}^{\varepsilon}(\boldsymbol{x}) = \sum_{n=1}^{N} \hat{\lambda}_n^{\varepsilon} \frac{G_\omega(\|\boldsymbol{x} - \boldsymbol{c}_n\|)}{G_\omega(r_n^{\varepsilon})},$$

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Rewriting in the time domain

$\boxed{\mathcal{G}(t,r) := \frac{1}{2\pi}\frac{H(t-r)}{\sqrt{t^2-r^2}}}$ 2D Green fn for the wave equation ($H(t)$ is Heaviside fn)

New unknown $\hat{\mu}_n^{\varepsilon} := \hat{\lambda}_n^{\varepsilon} G_\omega^{-1}(r_n^{\varepsilon})$

Foldy-Lax approximation

$$u_{FL}^{\varepsilon}(\boldsymbol{x},t) = \sum_{n=1}^{N} \int_0^t \mathcal{G}(t-\tau, \|\boldsymbol{x}-\boldsymbol{c}_n\|)\mu_n^{\varepsilon}(\tau)d\tau = \sum_{n=1}^{N} \mathcal{G}(t, \|\boldsymbol{x}-\boldsymbol{c}_n\|) * \mu_n^{\varepsilon}(t)$$

$$-u^{inc}(\boldsymbol{c}_k,t) = \mathcal{G}(t, r_k^{\varepsilon}) * \mu_k^{\varepsilon}(t) + \sum_{n\neq k} \mathcal{G}(t, \|\boldsymbol{c}_k-\boldsymbol{c}_n\|) * \mu_n^{\varepsilon}(t), \; k=1,\dots,N.$$

Numerical simulations

Stability? Convergence?

Uniform Stability: for a sufficiently regular u^{inc}, for all $\varepsilon_0 > \varepsilon > 0$, for all $t > 0$,

$$\|u^{\varepsilon}_{app}(t)\|_{L^2_{loc}} \leq C_{geom}(1+t)^q \|u^{inc}\|_{H^s(0,t;H^m(\mathbb{R}^2))}.$$

Convergence: $\|u^{\varepsilon}_{app} - u^{\varepsilon}\|/\|u^{\varepsilon}\| \to 0$, as $\varepsilon \to 0$.

Numerical simulations

We compare Foldy-Lax and FEM solutions (with the PML to bound the comp. domain)

Incident wave: $u^{inc} = -\mathrm{e}^{-20(t-\mathbf{x}\cdot\mathbf{d})^2}\sin(20(t-\mathbf{x}\cdot\mathbf{d})), \quad \mathbf{d} = (0,1)^T$

Obstacle diameter: $d = 0.01$ ($\sim 1/12\lambda$). **Final time:** $T = 4$

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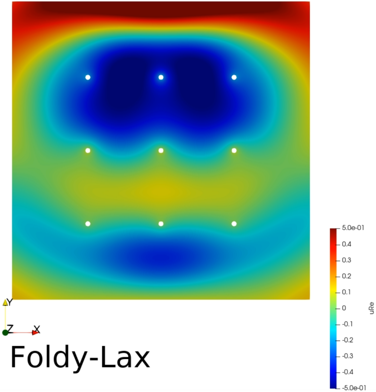

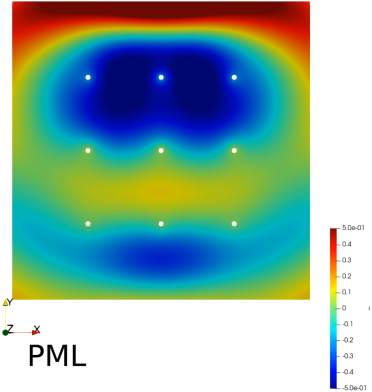

Numerical simulations

We compare Foldy-Lax and FEM solutions (with the PML to bound the comp. domain)

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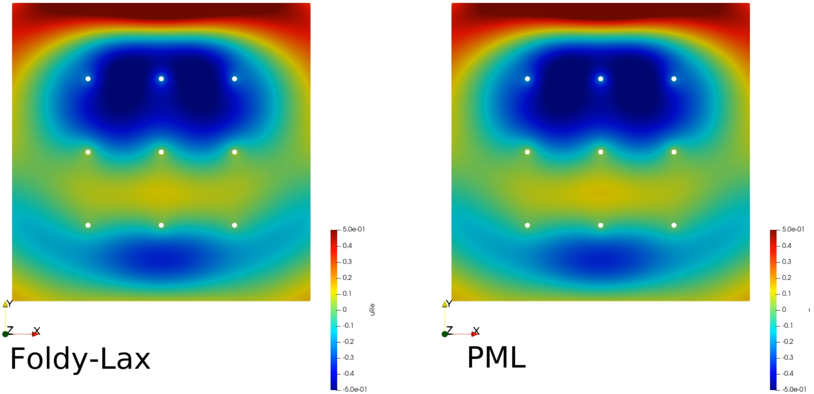

Seems to converge quite well !

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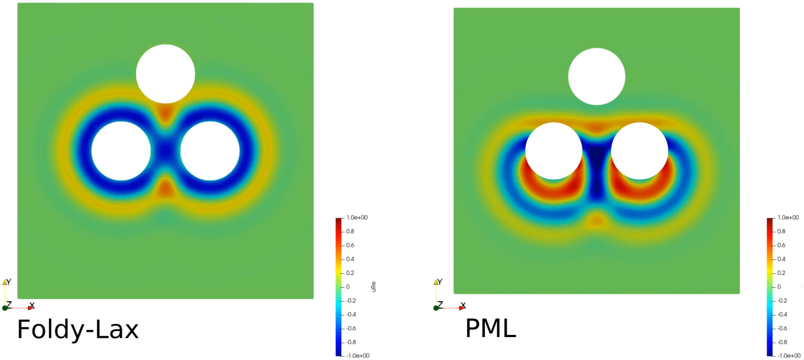

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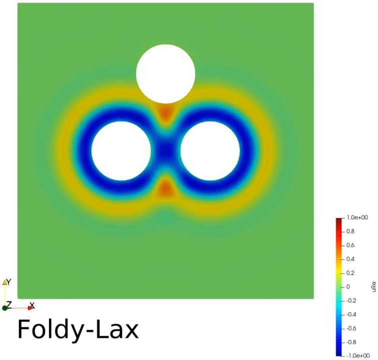
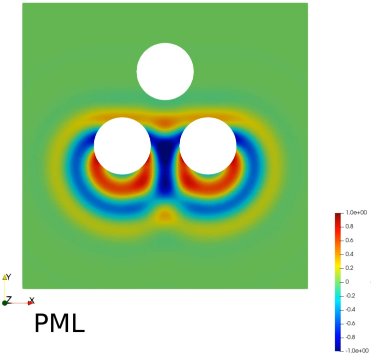
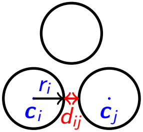




Exponential blow up (can be proven rigorously, occurs also in less exotic situations!)

An instability result



An example instability result

$N = 3$, $r_i = r$ and $0 < d_{ij} = \alpha r$, $\alpha < 2$, $\forall i, j$

For some $u^{inc} \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^2)$, some $t_n \to +\infty$, and $c_\lambda, A > 0$,

$$\|\boldsymbol{\lambda}(t_n)\| \geq c_\lambda \mathrm{e}^{A t_n}$$

(a similar result holds for $\boldsymbol{u}^\varepsilon_{app}$).

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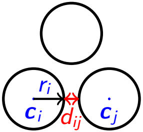

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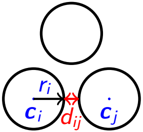

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Foldy-Lax equations

$$\begin{pmatrix} \hat{\lambda}^1 \\ \hat{\lambda}^2 \\ \hat{\lambda}^3 \end{pmatrix} = -\underbrace{\begin{pmatrix} 1 & P_\omega & P_\omega \\ P_\omega & 1 & P_\omega \\ P_\omega & P_\omega & 1 \end{pmatrix}^{-1}}_{M_\omega^{-1}} \begin{pmatrix} \hat{u}^{inc}(\boldsymbol{c}_1) \\ \hat{u}^{inc}(\boldsymbol{c}_2) \\ \hat{u}^{inc}(\boldsymbol{c}_3) \end{pmatrix}, \quad P_\omega := \frac{H_0^{(1)}((\alpha+2)\omega r)}{H_0^{(1)}(\omega r)}.$$

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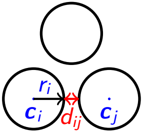

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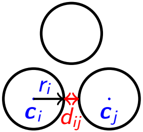

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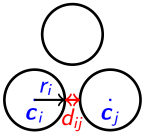

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Conclusion: for some geometries the FL model is unstable (lack of robustness)

Stabilization

Start with the boundary representations:

Single-layer representation $u^{\varepsilon}(t,\mathbf{x}) = \int_{\Gamma^{\varepsilon}} \int_0^t \mathcal{G}(t-\tau, \|\boldsymbol{x}-\boldsymbol{y}\|)\mu^{\varepsilon}(\tau,\boldsymbol{y})d\tau\, d\Gamma_y.$

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Outline

Outline of what follows

- derivation in the frequency domain
- passing to the time domain
- stability (will skip the statement itself) and convergence analysis
- some numerics

Galerkin Foldy-Lax model

Single Layer Ansatz: $\hat{u}^{\varepsilon}(\boldsymbol{x}) = \sum_{n=1}^{N} \int_{\Gamma_n^{\varepsilon}} G_{\omega}(\|\boldsymbol{x}-\boldsymbol{y}\|)\hat{\mu}_n^{\varepsilon}(\boldsymbol{y})d\Gamma_y = \mathcal{S}^{\varepsilon}\hat{\boldsymbol{\mu}}^{\varepsilon}, \quad \boldsymbol{x} \in \Omega^{\varepsilon,c}$

Single Layer BIE: given $\hat{\boldsymbol{g}}^{\varepsilon} \in H^{1/2}(\Gamma^{\varepsilon})$, find $\hat{\boldsymbol{\mu}}^{\varepsilon} \in H^{-1/2}(\Gamma^{\varepsilon})$, s.t.

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The Galerkin Foldy-Lax model in the frequency domain

Find $\hat{\boldsymbol{\mu}}_G^{\varepsilon} \in \mathbb{S}_0^{\varepsilon}$, s.t. for all $\phi \in \mathbb{S}_0^{\varepsilon}$,

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The Galerkin Foldy-Lax model more explicitly

(Taking $\phi = 1$ on Γ_k^{ε} and zero otherwise). Find $\hat{\boldsymbol{\mu}}_G^{\varepsilon} \in \mathbb{C}^N$, s.t.

$$\int_{\Gamma_k^{\varepsilon}} \hat{\boldsymbol{g}}^{\varepsilon}(\boldsymbol{x})d\Gamma_{\boldsymbol{x}} = \sum_{n=1}^{N} \hat{\mu}_{G,n}^{\varepsilon} \int_{\Gamma_k^{\varepsilon}} \int_{\Gamma_n^{\varepsilon}} G_\omega(\|\boldsymbol{x}-\boldsymbol{y}\|)d\Gamma_y \, d\Gamma_x, \quad k = 1, \ldots, N.$$

Passing from the frequency domain to the time domain

Frequency domain

Find $\hat{\boldsymbol{\mu}}_G^{\boldsymbol{\varepsilon}} \in \mathbb{C}^N$, s.t.

$$\int_{\Gamma_k^{\boldsymbol{\varepsilon}}} \hat{\boldsymbol{g}}^{\boldsymbol{\varepsilon}}(\boldsymbol{x}) d\Gamma_{\boldsymbol{x}} = \sum_{n=1}^{N} \hat{\mu}_{G,n}^{\boldsymbol{\varepsilon}} \int_{\Gamma_k^{\boldsymbol{\varepsilon}}} \int_{\Gamma_n^{\boldsymbol{\varepsilon}}} G_\omega(\|\boldsymbol{x}-\boldsymbol{y}\|) d\Gamma_y \, d\Gamma_x, \quad k = 1, \ldots, N.$$

The field is approximated as follows:

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Passing from the frequency domain to the time domain

Frequency domain

Find $\hat{\boldsymbol{\mu}}^{\varepsilon}_G \in \mathbb{C}^N$, s.t.

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Time-domain

Find $\boldsymbol{\mu}^{\varepsilon}_G \in C^{\ell}(0, T; \mathbb{R}^N)$, s.t.

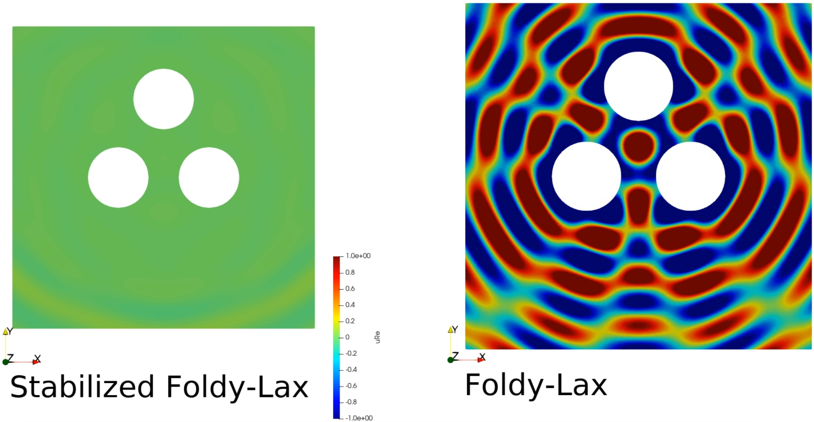
$$\int_{\Gamma^{\varepsilon}_k} g^{\varepsilon}_k(t, \boldsymbol{x}) d\Gamma_{\boldsymbol{x}} = \sum_{n=1}^{N} \int_0^t \underbrace{\left(\int_{\Gamma^{\varepsilon}_k} \int_{\Gamma^{\varepsilon}_n} \mathcal{G}(t-\tau, \|\boldsymbol{x}-\boldsymbol{y}\|) d\Gamma_y \, d\Gamma_x \right)}_{\mathcal{K}^{\varepsilon}_{kn}(t-\tau)} \mu^{\varepsilon}_{G,n}(\tau) d\tau, \quad k = 1, \dots, N.$$

The field then can be found by computing time-domain convolutions

$$u^{\varepsilon}_G(t, \boldsymbol{x}) = \sum_{n=1}^{N} \int_0^t \left(\int_{\Gamma^{\varepsilon}_n} \mathcal{G}(t-\tau, \|\boldsymbol{x}-\boldsymbol{y}\|) d\Gamma_{\boldsymbol{y}} \right) \mu^{\varepsilon}_{G,n}(\tau) d\tau, \quad \boldsymbol{x} \in \Omega^{\varepsilon,c}.$$

Stability

The previously 'unstable' configuration



Convergence analysis

What is a convergence order of the newly designed model?

Error analysis

Analysis: frequency domain ($\operatorname{Im} \omega > 0$ (ensures invertibility of all operators involved))+ explicit dependence on ω and ε+Plancherel $\implies$ time-domain

Error analysis

Galerkin Foldy-Lax method: Find $\hat{\mu}_G^\varepsilon \in \mathbb{S}_0^\varepsilon$, s.t. for all $\phi \in \mathbb{S}_0^\varepsilon$,

$$\langle \hat{g}^\varepsilon(\omega), \phi \rangle_{H^{1/2},H^{-1/2}} = \langle \mathbf{S}_\omega^\varepsilon \hat{\mu}_G^\varepsilon(\omega), \phi \rangle_{H^{1/2},H^{-1/2}} = \iint_{\Gamma^\varepsilon \times \Gamma^\varepsilon} G_\omega(\|x - y\|) \hat{\mu}_G^\varepsilon(x) \phi(y) d\Gamma_y.$$

$\hat{\mu}^\varepsilon$ satisfies the same but $\mathbb{S}_0^\varepsilon$ replaced by $H^{-1/2}(\Gamma^\varepsilon)$

Goal: estimate $\|\hat{\mu}_G^\varepsilon - \hat{\mu}^\varepsilon\|_{H^{-1/2}(\Gamma^\varepsilon)}$ ($\Longrightarrow$ an estimate on $\|u^\varepsilon - u_G^\varepsilon\|$)

Error analysis

Galerkin Foldy-Lax method: Find $\hat{\mu}_G^{\varepsilon} \in \mathbb{S}_0^{\varepsilon}$, s.t. for all $\phi \in \mathbb{S}_0^{\varepsilon}$,

$$\langle \hat{\mathbf{g}}^{\varepsilon}(\omega), \phi \rangle_{H^{1/2},H^{-1/2}} = \langle \mathbf{S}_{\omega}^{\varepsilon} \hat{\mu}_G^{\varepsilon}(\omega), \phi \rangle_{H^{1/2},H^{-1/2}} = \iint_{\Gamma^{\varepsilon} \times \Gamma^{\varepsilon}} G_{\omega}(\|\mathbf{x} - \mathbf{y}\|) \hat{\mu}_G^{\varepsilon}(\mathbf{x}) \phi(\mathbf{y}) d\Gamma_{\mathbf{y}}.$$

$\hat{\mu}^{\varepsilon}$ satisfies the same but $\mathbb{S}_0^{\varepsilon}$ replaced by $H^{-1/2}(\Gamma^{\varepsilon})$

Goal: estimate $\|\hat{\mu}_G^{\varepsilon} - \hat{\mu}^{\varepsilon}\|_{H^{-1/2}(\Gamma^{\varepsilon})}$ ($\Longrightarrow$ an estimate on $\|u^{\varepsilon} - u_G^{\varepsilon}\|$)

Error analysis

Galerkin Foldy-Lax method: Find $\hat{\mu}_G^\varepsilon \in \mathbb{S}_0^\varepsilon$, s.t. for all $\phi \in \mathbb{S}_0^\varepsilon$,

$$\langle \hat{\mathbf{g}}^\varepsilon(\omega), \phi \rangle_{H^{1/2}, H^{-1/2}} = \langle \mathbf{S}_\omega^\varepsilon \hat{\mu}_G^\varepsilon(\omega), \phi \rangle_{H^{1/2}, H^{-1/2}} = \iint_{\Gamma^\varepsilon \times \Gamma^\varepsilon} G_\omega(\|\mathbf{x} - \mathbf{y}\|) \hat{\mu}_G^\varepsilon(\mathbf{x}) \phi(\mathbf{y}) d\Gamma_{\mathbf{y}}.$$

$\hat{\mu}^\varepsilon$ satisfies the same but $\mathbb{S}_0^\varepsilon$ replaced by $H^{-1/2}(\Gamma^\varepsilon)$

Goal: estimate $\|\hat{\mu}_G^\varepsilon - \hat{\mu}^\varepsilon\|_{H^{-1/2}(\Gamma^\varepsilon)}$ ($\implies$ an estimate on $\|u^\varepsilon - u_G^\varepsilon\|$)

Decomposition of $H^{1/2}(\Gamma^\varepsilon)$:

$$H^{1/2}(\Gamma^\varepsilon) = \mathbb{S}_0^\varepsilon \overset{\perp_{L^2}}{+} H_*^{1/2}(\Gamma^\varepsilon),$$

$$\mathbb{S}_0^\varepsilon = \{\phi \in H^{1/2}(\Gamma^\varepsilon) : \ \phi|_{\Gamma_k^\varepsilon} = \text{const}\},$$

Error analysis

Galerkin Foldy-Lax method: Find $\hat{\mu}_G^\varepsilon \in \mathbb{S}_0^\varepsilon$, s.t. for all $\phi \in \mathbb{S}_0^\varepsilon$,

$$\langle \hat{\mathbf{g}}^\varepsilon(\omega), \phi \rangle_{H^{1/2},H^{-1/2}} = \langle \mathbf{S}_\omega^\varepsilon \hat{\mu}_G^\varepsilon(\omega), \phi \rangle_{H^{1/2},H^{-1/2}} = \iint_{\Gamma^\varepsilon \times \Gamma^\varepsilon} G_\omega(\|x - y\|)\hat{\mu}_G^\varepsilon(x)\phi(y) d\Gamma_y.$$

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$$H_*^{1/2}(\Gamma^\varepsilon) = \{\phi \in H^{1/2}(\Gamma^\varepsilon) : (\phi, \psi)_{L^2(\Gamma^\varepsilon)} = 0, \quad \forall \psi \in \mathbb{S}_0^\varepsilon\},$$

Error analysis

Galerkin Foldy-Lax method: Find $\hat{\boldsymbol{\mu}}_G^{\varepsilon} \in \mathbb{S}_0^{\varepsilon}$, s.t. for all $\phi \in \mathbb{S}_0^{\varepsilon}$,

$$\langle \hat{\boldsymbol{g}}^{\varepsilon}(\omega), \phi \rangle_{H^{1/2}, H^{-1/2}} = \langle \mathbf{S}_{\omega}^{\varepsilon} \hat{\boldsymbol{\mu}}_G^{\varepsilon}(\omega), \phi \rangle_{H^{1/2}, H^{-1/2}} = \iint_{\Gamma^{\varepsilon} \times \Gamma^{\varepsilon}} G_{\omega}(\|\boldsymbol{x} - \boldsymbol{y}\|) \hat{\mu}_G^{\varepsilon}(\boldsymbol{x}) \phi(\boldsymbol{y}) d\Gamma_{\boldsymbol{y}}.$$

$\hat{\boldsymbol{\mu}}^{\varepsilon}$ satisfies the same but $\mathbb{S}_0^{\varepsilon}$ replaced by $H^{-1/2}(\Gamma^{\varepsilon})$

Goal: estimate $\|\hat{\boldsymbol{\mu}}_G^{\varepsilon} - \hat{\boldsymbol{\mu}}^{\varepsilon}\|_{H^{-1/2}(\Gamma^{\varepsilon})}$ ($\Longrightarrow$ an estimate on $\|u^{\varepsilon} - u_G^{\varepsilon}\|$)

Decomposition of $H^{1/2}(\Gamma^{\varepsilon})$:

$$H^{1/2}(\Gamma^{\varepsilon}) = \mathbb{S}_0^{\varepsilon} \overset{\perp_{L^2}}{+} H_*^{1/2}(\Gamma^{\varepsilon}),$$

$$\mathbb{S}_0^{\varepsilon} = \{\phi \in H^{1/2}(\Gamma^{\varepsilon}) : \ \phi|_{\Gamma_k^{\varepsilon}} = \text{const}\},$$

$$H_*^{1/2}(\Gamma^{\varepsilon}) = \{\phi \in H^{1/2}(\Gamma^{\varepsilon}) : (\phi, \psi)_{L^2(\Gamma^{\varepsilon})} = 0, \quad \forall \psi \in \mathbb{S}_0^{\varepsilon}\},$$

$\mathbb{P}_0, \mathbb{P}_{\perp}$ resp. orthog. projectors ($\Longleftrightarrow$ $v = \mathbb{P}_0 v + \mathbb{P}_{\perp} v = v_0 + v_{\perp}$)

Error analysis

Galerkin Foldy-Lax method: Find $\hat{\mu}_G^\varepsilon \in \mathbb{S}_0^\varepsilon$, s.t. for all $\phi \in \mathbb{S}_0^\varepsilon$,

$$\langle \hat{\mathbf{g}}^\varepsilon(\omega), \phi \rangle_{H^{1/2},H^{-1/2}} = \langle \mathbf{S}_\omega^\varepsilon \hat{\mu}_G^\varepsilon(\omega), \phi \rangle_{H^{1/2},H^{-1/2}} = \iint_{\Gamma^\varepsilon \times \Gamma^\varepsilon} G_\omega(\|\mathbf{x}-\mathbf{y}\|)\hat{\mu}_G^\varepsilon(\mathbf{x})\phi(\mathbf{y})d\Gamma_{\mathbf{y}}.$$

$\hat{\mu}^\varepsilon$ satisfies the same but $\mathbb{S}_0^\varepsilon$ replaced by $H^{-1/2}(\Gamma^\varepsilon)$

Goal: estimate $\|\hat{\mu}_G^\varepsilon - \hat{\mu}^\varepsilon\|_{H^{-1/2}(\Gamma^\varepsilon)}$ ($\Longrightarrow$ an estimate on $\|u^\varepsilon - u_G^\varepsilon\|$)

Decomposition of $H^{1/2}(\Gamma^\varepsilon)$: $H^{1/2}(\Gamma^\varepsilon) = \mathbb{S}_0^\varepsilon \overset{\perp_{L^2}}{+} H_*^{1/2}(\Gamma^\varepsilon)$,

$$\mathbb{S}_0^\varepsilon = \{\phi \in H^{1/2}(\Gamma^\varepsilon) : \ \phi|_{\Gamma_k^\varepsilon} = \text{const}\},$$

$$H_*^{1/2}(\Gamma^\varepsilon) = \{\phi \in H^{1/2}(\Gamma^\varepsilon) : (\phi, \psi)_{L^2(\Gamma^\varepsilon)} = 0, \quad \forall \psi \in \mathbb{S}_0^\varepsilon\},$$

$\mathbb{P}_0, \mathbb{P}_\perp$ resp. orthog. projectors ($\Longleftrightarrow$ $v = \mathbb{P}_0 v + \mathbb{P}_\perp v = v_0 + v_\perp$)

Decomposition of $H^{-1/2}(\Gamma^\varepsilon)$: $H^{-1/2}(\Gamma^\varepsilon) = \mathbb{S}_0^\varepsilon \dot{+} H_*^{-1/2}(\Gamma^\varepsilon)$

$$H_*^{-1/2}(\Gamma^\varepsilon) = \{\phi \in H^{-1/2}(\Gamma^\varepsilon) : \langle \phi, \psi \rangle_{H^{-1/2},H^{1/2}} = 0, \quad \forall \psi \in \mathbb{S}_0^\varepsilon\},$$

$$v = \mathbb{P}_0^* v + \mathbb{P}_\perp^* v = v_0 + v_\perp.$$

Error analysis

Galerkin Foldy-Lax method: Find $\hat{\mu}^\varepsilon_G \in \mathbb{S}^\varepsilon_0$, s.t. for all $\phi \in \mathbb{S}^\varepsilon_0$,

$$\langle \hat{g}^\varepsilon(\omega), \phi \rangle_{H^{1/2},H^{-1/2}} = \langle \mathbf{S}^\varepsilon_\omega \hat{\mu}^\varepsilon_G(\omega), \phi \rangle_{H^{1/2},H^{-1/2}} = \iint_{\Gamma^\varepsilon \times \Gamma^\varepsilon} G_\omega(\|x - y\|)\hat{\mu}^\varepsilon_G(x)\phi(y)d\Gamma_y.$$

$\hat{\mu}^\varepsilon$ satisfies the same but $\mathbb{S}^\varepsilon_0$ replaced by $H^{-1/2}(\Gamma^\varepsilon)$

Goal: estimate $\|\hat{\mu}^\varepsilon_G - \hat{\mu}^\varepsilon\|_{H^{-1/2}(\Gamma^\varepsilon)}$ ($\Longrightarrow$ an estimate on $\|u^\varepsilon - u^\varepsilon_G\|$)

Decomposition of $H^{1/2}(\Gamma^\varepsilon)$: $H^{1/2}(\Gamma^\varepsilon) = \mathbb{S}^\varepsilon_0 \overset{\perp_{L^2}}{+} H^{1/2}_*(\Gamma^\varepsilon),$

$$\mathbb{S}^\varepsilon_0 = \{\phi \in H^{1/2}(\Gamma^\varepsilon) : \ \phi|_{\Gamma^\varepsilon_k} = \text{const}\},$$

$$H^{1/2}_*(\Gamma^\varepsilon) = \{\phi \in H^{1/2}(\Gamma^\varepsilon) : (\phi, \psi)_{L^2(\Gamma^\varepsilon)} = 0, \quad \forall \psi \in \mathbb{S}^\varepsilon_0\},$$

$\mathbb{P}_0, \mathbb{P}_\perp$ resp. orthog. projectors ($\Longleftrightarrow$ $v = \mathbb{P}_0 v + \mathbb{P}_\perp v = v_0 + v_\perp$)

Decomposition of $H^{-1/2}(\Gamma^\varepsilon)$: $H^{-1/2}(\Gamma^\varepsilon) = \mathbb{S}^\varepsilon_0 \dot{+} H^{-1/2}_*(\Gamma^\varepsilon)$

$$H^{-1/2}_*(\Gamma^\varepsilon) = \{\phi \in H^{-1/2}(\Gamma^\varepsilon) : \langle \phi, \psi \rangle_{H^{-1/2},H^{1/2}} = 0, \quad \forall \psi \in \mathbb{S}^\varepsilon_0\},$$

$$v = \mathbb{P}^*_0 v + \mathbb{P}^*_\perp v = v_0 + v_\perp.$$

Coercivity/continuity constants of $\mathbf{S}^\varepsilon_\omega$ on different spaces have a different asymptotic behavior w.r.t. ε (seen e.g. from a scaling argument for one obstacle)

Error analysis

Galerkin Foldy-Lax problem: Find $\hat{\boldsymbol{\mu}}_G^{\varepsilon} \in \mathbb{S}_0^{\varepsilon}$, s.t. for all $\phi \in \mathbb{S}_0^{\varepsilon}$,

$$\langle \hat{\boldsymbol{g}}^{\varepsilon}(\omega), \phi \rangle = \langle \mathbf{S}^{\varepsilon} \hat{\boldsymbol{\mu}}_G^{\varepsilon}(\omega), \phi \rangle. \tag{GD}$$

The error: $\hat{\mathbf{e}}^{\varepsilon} = \hat{\boldsymbol{\mu}}^{\varepsilon} - \boldsymbol{\mu}_G^{\varepsilon} = \hat{\mathbf{e}}_0^{\varepsilon} + \hat{\mathbf{e}}_{\perp}^{\varepsilon}$

Error analysis

Galerkin Foldy-Lax problem: Find $\hat{\boldsymbol{\mu}}_G^\varepsilon \in \mathbb{S}_0^\varepsilon$, s.t. for all $\phi \in \mathbb{S}_0^\varepsilon$,

$$\langle \hat{\boldsymbol{g}}^\varepsilon(\omega), \phi \rangle = \langle \mathbf{S}^\varepsilon \hat{\boldsymbol{\mu}}_G^\varepsilon(\omega), \phi \rangle. \tag{GD}$$

The error: $\hat{\boldsymbol{e}}^\varepsilon = \hat{\boldsymbol{\mu}}^\varepsilon - \boldsymbol{\mu}_G^\varepsilon = \hat{\boldsymbol{e}}_0^\varepsilon + \hat{\boldsymbol{e}}_\perp^\varepsilon$

Goal: Obtain a bound on $\hat{\boldsymbol{e}^\varepsilon}$ in terms of the data ($\hat{u}^{inc}$)

Error analysis

Galerkin Foldy-Lax problem: Find $\hat{\mu}_G^\varepsilon \in \mathbb{S}_0^\varepsilon$, s.t. for all $\phi \in \mathbb{S}_0^\varepsilon$,

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The error: $\hat{\boldsymbol{e}}^\varepsilon = \hat{\mu}^\varepsilon - \mu_G^\varepsilon = \hat{\boldsymbol{e}}_0^\varepsilon + \hat{\boldsymbol{e}}_\perp^\varepsilon$

Goal: Obtain a bound on $\hat{\boldsymbol{e}^\varepsilon}$ in terms of the data ($\hat{u}^{inc}$)

Equation for the error

$$\mathbf{S}_{00}^\varepsilon := \mathbb{P}_0 \mathbf{S}^\varepsilon \mathbb{P}_0^*, \quad \mathbf{S}_{\perp\perp}^\varepsilon := \mathbb{P}_\perp \mathbf{S}^\varepsilon \mathbb{P}_\perp^*$$

Error analysis

Galerkin Foldy-Lax problem: Find $\hat{\mu}_G^\varepsilon \in \mathbb{S}_0^\varepsilon$, s.t. for all $\phi \in \mathbb{S}_0^\varepsilon$,

$$\langle \hat{\boldsymbol{g}}^\varepsilon(\omega), \phi \rangle = \langle \mathbf{S}^\varepsilon \hat{\mu}_G^\varepsilon(\omega), \phi \rangle. \qquad \text{(GD)}$$

The error: $\hat{\boldsymbol{e}}^\varepsilon = \hat{\mu}^\varepsilon - \mu_G^\varepsilon = \hat{\boldsymbol{e}}_0^\varepsilon + \hat{\boldsymbol{e}}_\perp^\varepsilon$

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Equation for the error

$$\mathbf{S}_{00}^\varepsilon := \mathbb{P}_0 \mathbf{S}^\varepsilon \mathbb{P}_0^*, \quad \mathbf{S}_{\perp\perp}^\varepsilon := \mathbb{P}_\perp \mathbf{S}^\varepsilon \mathbb{P}_\perp^*,$$
$$\mathbf{S}_{\perp 0}^\varepsilon := \mathbb{P}_\perp \mathbf{S}^\varepsilon \mathbb{P}_0^*, \quad \mathbf{S}_{0\perp}^\varepsilon = \mathbb{P}_0 \mathbf{S}^\varepsilon \mathbb{P}_\perp^*.$$

Same notation for operators restricted to the corresp. spaces $\mathbf{S}_{00}^\varepsilon = \mathbb{P}_0 \mathbf{S}^\varepsilon|_{\mathbb{S}_0^\varepsilon}$

Error analysis

Galerkin Foldy-Lax problem: Find $\hat{\mu}_G^\varepsilon \in \mathbb{S}_0^\varepsilon$, s.t. for all $\phi \in \mathbb{S}_0^\varepsilon$,

$$\langle \hat{\mathbf{g}}^\varepsilon(\omega), \phi \rangle = \langle \mathbf{S}^\varepsilon \hat{\mu}_G^\varepsilon(\omega), \phi \rangle. \tag{GD}$$

The error: $\hat{\mathbf{e}}^\varepsilon = \hat{\mu}^\varepsilon - \mu_G^\varepsilon = \hat{\mathbf{e}}_0^\varepsilon + \hat{\mathbf{e}}_\perp^\varepsilon$

Goal: Obtain a bound on $\hat{\mathbf{e}^\varepsilon}$ in terms of the data ($\hat{u}^{inc}$)

Equation for the error

$$\mathbf{S}_{00}^\varepsilon := \mathbb{P}_0 \mathbf{S}^\varepsilon \mathbb{P}_0^*, \quad \mathbf{S}_{\perp\perp}^\varepsilon := \mathbb{P}_\perp \mathbf{S}^\varepsilon \mathbb{P}_\perp^*,$$
$$\mathbf{S}_{\perp 0}^\varepsilon := \mathbb{P}_\perp \mathbf{S}^\varepsilon \mathbb{P}_0^*, \quad \mathbf{S}_{0\perp}^\varepsilon = \mathbb{P}_0 \mathbf{S}^\varepsilon \mathbb{P}_\perp^*.$$

Same notation for operators restricted to the corresp. spaces $\mathbf{S}_{00}^\varepsilon = \mathbb{P}_0 \mathbf{S}^\varepsilon|_{\mathbb{S}_0^\varepsilon}$

Galerkin Foldy-Lax problem: $\mathbf{S}_{00}^\varepsilon \mu_G^\varepsilon = \hat{\mathbf{g}}_0^\varepsilon$

Error analysis

Galerkin Foldy-Lax problem: Find $\hat{\mu}_G^\varepsilon \in \mathbb{S}_0^\varepsilon$, s.t. for all $\phi \in \mathbb{S}_0^\varepsilon$,

$$\langle \hat{\boldsymbol{g}}^\varepsilon(\omega), \phi \rangle = \langle \mathbf{S}^\varepsilon \hat{\mu}_G^\varepsilon(\omega), \phi \rangle. \tag{GD}$$

The error: $\hat{\boldsymbol{e}}^\varepsilon = \hat{\mu}^\varepsilon - \mu_G^\varepsilon = \hat{\boldsymbol{e}}_0^\varepsilon + \hat{\boldsymbol{e}}_\perp^\varepsilon$

Goal: Obtain a bound on $\hat{\boldsymbol{e}^\varepsilon}$ in terms of the data ($\hat{u}^{inc}$)

Equation for the error

$$\mathbf{S}_{00}^\varepsilon := \mathbb{P}_0 \mathbf{S}^\varepsilon \mathbb{P}_0^*, \quad \mathbf{S}_{\perp\perp}^\varepsilon := \mathbb{P}_\perp \mathbf{S}^\varepsilon \mathbb{P}_\perp^*,$$
$$\mathbf{S}_{\perp 0}^\varepsilon := \mathbb{P}_\perp \mathbf{S}^\varepsilon \mathbb{P}_0^*, \quad \mathbf{S}_{0\perp}^\varepsilon = \mathbb{P}_0 \mathbf{S}^\varepsilon \mathbb{P}_\perp^*.$$

Same notation for operators restricted to the corresp. spaces $\mathbf{S}_{00}^\varepsilon = \mathbb{P}_0 \mathbf{S}^\varepsilon |_{\mathbb{S}_0^\varepsilon}$

Galerkin Foldy-Lax problem: $\mathbf{S}_{00}^\varepsilon \mu_G^\varepsilon = \hat{\boldsymbol{g}}_0^\varepsilon$

Exact problem: $\begin{pmatrix} \mathbf{S}_{00}^\varepsilon & \mathbf{S}_{0\perp}^\varepsilon \\ \mathbf{S}_{\perp 0}^\varepsilon & \mathbf{S}_{\perp\perp}^\varepsilon \end{pmatrix} \begin{pmatrix} \hat{\mu}_0^\varepsilon \\ \hat{\mu}_\perp^\varepsilon \end{pmatrix} = \begin{pmatrix} \hat{\boldsymbol{g}}_0^\varepsilon \\ \hat{\boldsymbol{g}}_\perp^\varepsilon \end{pmatrix}.$

Error analysis

Galerkin Foldy-Lax problem: Find $\hat{\mu}_G^\varepsilon \in \mathbb{S}_0^\varepsilon$, s.t. for all $\phi \in \mathbb{S}_0^\varepsilon$,

$$\langle \hat{\boldsymbol{g}}^\varepsilon(\omega), \phi \rangle = \langle \mathbf{S}^\varepsilon \hat{\mu}_G^\varepsilon(\omega), \phi \rangle. \qquad \text{(GD)}$$

The error: $\hat{\boldsymbol{e}}^\varepsilon = \hat{\mu}^\varepsilon - \mu_G^\varepsilon = \hat{\boldsymbol{e}}_0^\varepsilon + \hat{\boldsymbol{e}}_\perp^\varepsilon$

Goal: Obtain a bound on $\hat{\boldsymbol{e}^\varepsilon}$ in terms of the data ($\hat{u}^{inc}$)

Equation for the error

$$\mathbf{S}_{00}^\varepsilon := \mathbb{P}_0 \mathbf{S}^\varepsilon \mathbb{P}_0^*, \quad \mathbf{S}_{\perp\perp}^\varepsilon := \mathbb{P}_\perp \mathbf{S}^\varepsilon \mathbb{P}_\perp^*,$$
$$\mathbf{S}_{\perp 0}^\varepsilon := \mathbb{P}_\perp \mathbf{S}^\varepsilon \mathbb{P}_0^*, \quad \mathbf{S}_{0\perp}^\varepsilon = \mathbb{P}_0 \mathbf{S}^\varepsilon \mathbb{P}_\perp^*.$$

Same notation for operators restricted to the corresp. spaces $\mathbf{S}_{00}^\varepsilon = \mathbb{P}_0 \mathbf{S}^\varepsilon|_{\mathbb{S}_0^\varepsilon}$

Galerkin Foldy-Lax problem: $\mathbf{S}_{00}^\varepsilon \mu_G^\varepsilon = \hat{\boldsymbol{g}}_0^\varepsilon$

Exact problem: $\begin{pmatrix} \mathbf{S}_{00}^\varepsilon & \mathbf{S}_{0\perp}^\varepsilon \\ \mathbf{S}_{\perp 0}^\varepsilon & \mathbf{S}_{\perp\perp}^\varepsilon \end{pmatrix} \begin{pmatrix} \hat{\mu}_0^\varepsilon \\ \hat{\mu}_\perp^\varepsilon \end{pmatrix} = \begin{pmatrix} \hat{\boldsymbol{g}}_0^\varepsilon \\ \hat{\boldsymbol{g}}_\perp^\varepsilon \end{pmatrix}.$

The problem satisfied by the error:

$$\begin{pmatrix} \mathbf{S}_{00}^\varepsilon & \mathbf{S}_{0\perp}^\varepsilon \\ \mathbf{S}_{\perp 0}^\varepsilon & \mathbf{S}_{\perp\perp}^\varepsilon \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{e}}_0^\varepsilon \\ \hat{\boldsymbol{e}}_\perp^\varepsilon \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{\boldsymbol{g}}_\perp^\varepsilon - \mathbf{S}_{\perp 0}^\varepsilon \hat{\mu}_G^\varepsilon \end{pmatrix}.$$

Error analysis

$$\begin{pmatrix} \mathbf{S}_{00}^{\varepsilon} & \mathbf{S}_{0\perp}^{\varepsilon} \\ \mathbf{S}_{\perp 0}^{\varepsilon} & \mathbf{S}_{\perp\perp}^{\varepsilon} \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{e}}_{0}^{\varepsilon} \\ \hat{\boldsymbol{e}}_{\perp}^{\varepsilon} \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{\boldsymbol{g}}_{\perp}^{\varepsilon} - \mathbf{S}_{\perp 0}^{\varepsilon} \hat{\boldsymbol{\mu}}_{G}^{\varepsilon} \end{pmatrix}.$$

Error analysis

$$\begin{pmatrix} \mathbf{S}^{\varepsilon}_{00} & \mathbf{S}^{\varepsilon}_{0\perp} \\ \mathbf{S}^{\varepsilon}_{\perp 0} & \mathbf{S}^{\varepsilon}_{\perp\perp} \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{e}}^{\varepsilon}_{0} \\ \hat{\boldsymbol{e}}^{\varepsilon}_{\perp} \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{\boldsymbol{g}}^{\varepsilon}_{\perp} - \mathbf{S}^{\varepsilon}_{\perp 0} \hat{\boldsymbol{\mu}}^{\varepsilon}_{G} \end{pmatrix}.$$

$$\hat{\boldsymbol{e}}^{\varepsilon}_{0} = -(\mathbf{S}^{\varepsilon}_{00})^{-1} \mathbf{S}^{\varepsilon}_{0\perp} \hat{\boldsymbol{e}}^{\varepsilon}_{\perp},$$

Error analysis

$$\begin{pmatrix} \mathbf{S}^{\varepsilon}_{00} & \mathbf{S}^{\varepsilon}_{0\perp} \\ \mathbf{S}^{\varepsilon}_{\perp 0} & \mathbf{S}^{\varepsilon}_{\perp\perp} \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{e}}^{\varepsilon}_{0} \\ \hat{\boldsymbol{e}}^{\varepsilon}_{\perp} \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{\boldsymbol{g}}^{\varepsilon}_{\perp} - \mathbf{S}^{\varepsilon}_{\perp 0} \hat{\boldsymbol{\mu}}^{\varepsilon}_{G} \end{pmatrix}.$$

$$\hat{\boldsymbol{e}}^{\varepsilon}_{0} = -(\mathbf{S}^{\varepsilon}_{00})^{-1} \mathbf{S}^{\varepsilon}_{0\perp} \hat{\boldsymbol{e}}^{\varepsilon}_{\perp},$$

$$\hat{\boldsymbol{e}}^{\varepsilon}_{\perp} = \mathbb{P}^{*}_{\perp} (\mathbf{S}^{\varepsilon})^{-1} \mathbb{P}_{\perp} (\hat{\boldsymbol{g}}^{\varepsilon}_{\perp} - \mathbf{S}^{\varepsilon}_{\perp 0} \hat{\boldsymbol{\mu}}^{\varepsilon}_{G})$$

Error analysis

$$\begin{pmatrix} \mathbf{S}^{\varepsilon}_{00} & \mathbf{S}^{\varepsilon}_{0\perp} \\ \mathbf{S}^{\varepsilon}_{\perp 0} & \mathbf{S}^{\varepsilon}_{\perp\perp} \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{e}}^{\varepsilon}_{0} \\ \hat{\boldsymbol{e}}^{\varepsilon}_{\perp} \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{\boldsymbol{g}}^{\varepsilon}_{\perp} - \mathbf{S}^{\varepsilon}_{\perp 0} \hat{\boldsymbol{\mu}}^{\varepsilon}_{G} \end{pmatrix}.$$

$$\hat{\boldsymbol{e}}^{\varepsilon}_{0} = -(\mathbf{S}^{\varepsilon}_{00})^{-1} \mathbf{S}^{\varepsilon}_{0\perp} \hat{\boldsymbol{e}}^{\varepsilon}_{\perp},$$

$$\hat{\boldsymbol{e}}^{\varepsilon}_{\perp} = \mathbb{P}^{*}_{\perp} (\mathbf{S}^{\varepsilon})^{-1} \mathbb{P}_{\perp} (\hat{\boldsymbol{g}}^{\varepsilon}_{\perp} - \mathbf{S}^{\varepsilon}_{\perp 0} \hat{\boldsymbol{\mu}}^{\varepsilon}_{G}) = \mathbb{P}^{*}_{\perp} (\mathbf{S}^{\varepsilon})^{-1} \mathbb{P}_{\perp} \left(\hat{\boldsymbol{g}}^{\varepsilon}_{\perp} - \mathbf{S}^{\varepsilon}_{\perp 0} (\mathbf{S}^{\varepsilon}_{00})^{-1} \hat{\boldsymbol{g}}^{\varepsilon}_{0} \right)$$

Error analysis

$$\begin{pmatrix} \mathbf{S}^{\varepsilon}_{00} & \mathbf{S}^{\varepsilon}_{0\perp} \\ \mathbf{S}^{\varepsilon}_{\perp 0} & \mathbf{S}^{\varepsilon}_{\perp\perp} \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{e}}^{\varepsilon}_{0} \\ \hat{\boldsymbol{e}}^{\varepsilon}_{\perp} \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{\boldsymbol{g}}^{\varepsilon}_{\perp} - \mathbf{S}^{\varepsilon}_{\perp 0}\hat{\boldsymbol{\mu}}^{\varepsilon}_{G} \end{pmatrix}.$$

$$\hat{\boldsymbol{e}}^{\varepsilon}_{0} = -(\mathbf{S}^{\varepsilon}_{00})^{-1}\mathbf{S}^{\varepsilon}_{0\perp}\hat{\boldsymbol{e}}^{\varepsilon}_{\perp},$$

$$\hat{\boldsymbol{e}}^{\varepsilon}_{\perp} = \mathbb{P}^{*}_{\perp}(\mathbf{S}^{\varepsilon})^{-1}\mathbb{P}_{\perp}\left(\hat{\boldsymbol{g}}^{\varepsilon}_{\perp} - \mathbf{S}^{\varepsilon}_{\perp 0}\hat{\boldsymbol{\mu}}^{\varepsilon}_{G}\right) = \mathbb{P}^{*}_{\perp}(\mathbf{S}^{\varepsilon})^{-1}\mathbb{P}_{\perp}\left(\hat{\boldsymbol{g}}^{\varepsilon}_{\perp} - \mathbf{S}^{\varepsilon}_{\perp 0}(\mathbf{S}^{\varepsilon}_{00})^{-1}\hat{\boldsymbol{g}}^{\varepsilon}_{0}\right)$$

Conclusion

$$\|\hat{\boldsymbol{e}}^{\varepsilon}_{\perp}\|_{H^{-1/2}_{*}} \leq \|\mathbb{P}^{*}_{\perp}(\mathbf{S}^{\varepsilon})^{-1}\mathbb{P}_{\perp}\| \left(\|\hat{\boldsymbol{g}}^{\varepsilon}_{\perp}\|_{H^{1/2}_{*}} + \|\mathbf{S}^{\varepsilon}_{\perp 0}\|\|(\mathbf{S}^{\varepsilon}_{00})^{-1}\|\|\hat{\boldsymbol{g}}^{\varepsilon}_{0}\|_{L^2}\right),$$

$$\|\hat{\boldsymbol{e}}^{\varepsilon}_{0}\|_{L^2} \leq \|(\mathbf{S}^{\varepsilon}_{00})^{-1}\|\|\mathbf{S}^{\varepsilon}_{0\perp}\|\|\hat{\boldsymbol{e}}^{\varepsilon}_{\perp}\|_{H^{-1/2}_{*}}$$

Error analysis

$$\begin{pmatrix} \mathbf{S}^{\varepsilon}_{00} & \mathbf{S}^{\varepsilon}_{0\perp} \\ \mathbf{S}^{\varepsilon}_{\perp 0} & \mathbf{S}^{\varepsilon}_{\perp\perp} \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{e}}^{\varepsilon}_{0} \\ \hat{\boldsymbol{e}}^{\varepsilon}_{\perp} \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{\boldsymbol{g}}^{\varepsilon}_{\perp} - \mathbf{S}^{\varepsilon}_{\perp 0}\hat{\boldsymbol{\mu}}^{\varepsilon}_{G} \end{pmatrix}.$$

$$\hat{\boldsymbol{e}}^{\varepsilon}_{0} = -(\mathbf{S}^{\varepsilon}_{00})^{-1}\mathbf{S}^{\varepsilon}_{0\perp}\hat{\boldsymbol{e}}^{\varepsilon}_{\perp},$$

$$\hat{\boldsymbol{e}}^{\varepsilon}_{\perp} = \mathbb{P}^{*}_{\perp}(\mathbf{S}^{\varepsilon})^{-1}\mathbb{P}_{\perp}\left(\hat{\boldsymbol{g}}^{\varepsilon}_{\perp} - \mathbf{S}^{\varepsilon}_{\perp 0}\hat{\boldsymbol{\mu}}^{\varepsilon}_{G}\right) = \mathbb{P}^{*}_{\perp}(\mathbf{S}^{\varepsilon})^{-1}\mathbb{P}_{\perp}\left(\hat{\boldsymbol{g}}^{\varepsilon}_{\perp} - \mathbf{S}^{\varepsilon}_{\perp 0}(\mathbf{S}^{\varepsilon}_{00})^{-1}\hat{\boldsymbol{g}}^{\varepsilon}_{0}\right)$$

Conclusion

$$\|\hat{\boldsymbol{e}}^{\varepsilon}_{\perp}\|_{H^{-1/2}_{*}} \leq \|\mathbb{P}^{*}_{\perp}(\mathbf{S}^{\varepsilon})^{-1}\mathbb{P}_{\perp}\|\left(\|\hat{\boldsymbol{g}}^{\varepsilon}_{\perp}\|_{H^{1/2}_{*}} + \|\mathbf{S}^{\varepsilon}_{\perp 0}\|\|(\mathbf{S}^{\varepsilon}_{00})^{-1}\|\|\hat{\boldsymbol{g}}^{\varepsilon}_{0}\|_{L^2}\right),$$

$$\|\hat{\boldsymbol{e}}^{\varepsilon}_{0}\|_{L^2} \leq \|(\mathbf{S}^{\varepsilon}_{00})^{-1}\|\|\mathbf{S}^{\varepsilon}_{0\perp}\|\|\hat{\boldsymbol{e}}^{\varepsilon}_{\perp}\|_{H^{-1/2}_{*}}$$

Convergence

Convergence=bounds on the operators+bounds on the data $\boldsymbol{g}^{\varepsilon} = -\, u^{inc}\big|_{\Gamma^{\varepsilon}}$

Error analysis

Conclusion

$$\|\hat{\boldsymbol{e}}_\perp^{\boldsymbol{\varepsilon}}\|_{H_*^{-1/2}} \leq \|\mathbb{P}_\perp^*(\mathbf{S}^{\boldsymbol{\varepsilon}})^{-1}\mathbb{P}_\perp\| \left(\|\hat{\boldsymbol{g}}_\perp^{\boldsymbol{\varepsilon}}\|_{H_*^{1/2}} + \|\mathbf{S}_{\perp 0}^{\boldsymbol{\varepsilon}}\|\|(\mathbf{S}_{00}^{\varepsilon})^{-1}\|\|\hat{\boldsymbol{g}}_0^{\boldsymbol{\varepsilon}}\|_{L^2}\right),$$

$$\|\hat{\boldsymbol{e}}_0^{\boldsymbol{\varepsilon}}\|_{L^2} \leq \|(\mathbf{S}_{00}^{\varepsilon})^{-1}\|\|\mathbf{S}_{0\perp}^{\boldsymbol{\varepsilon}}\|\|\hat{\boldsymbol{e}}_\perp^{\boldsymbol{\varepsilon}}\|_{H_*^{-1/2}}$$

Bounds on the operators (following ideas of Hassan, Stamm '21)

With constants that depend on ω, N and d_{ij},

$$\|(\mathbf{S}_{00}^{\boldsymbol{\varepsilon}})^{-1}\| \lesssim \varepsilon^{-1}, \quad \|\mathbb{P}_\perp^*(\mathbf{S}^{\boldsymbol{\varepsilon}})^{-1}\mathbb{P}_\perp\| \lesssim 1, \quad \|\mathbf{S}_{\perp 0}^{\boldsymbol{\varepsilon}}\| = \|\mathbf{S}_{0\perp}^{\boldsymbol{\varepsilon}}\| \lesssim \varepsilon^{3/2}$$

Error analysis

Conclusion

$$\|\hat{\boldsymbol{e}}_\perp\|_{H^{1/2}} \lesssim \left(\|\hat{\boldsymbol{g}}^{\boldsymbol{\varepsilon}}_\perp\| + \varepsilon^{1/2}\|\hat{\boldsymbol{g}}^{\boldsymbol{\varepsilon}}_0\|_{L^2}\right),$$

$$\|\hat{\boldsymbol{e}}_0\|_{L^2} \lesssim \varepsilon^{1/2}\|\hat{\boldsymbol{e}}_\perp\|_{H^{-1/2}}.$$

Bounds on the operators (following ideas of Hassan, Stamm '21)

With constants that depend on ω, N and d_{ij},

$$\|(\mathbf{S}^{\boldsymbol{\varepsilon}}_{00})^{-1}\| \lesssim \varepsilon^{-1}, \quad \|\mathbb{P}^*_\perp(\mathbf{S}^{\boldsymbol{\varepsilon}})^{-1}\mathbb{P}_\perp\| \lesssim 1, \quad \|\mathbf{S}^{\boldsymbol{\varepsilon}}_{\perp 0}\| = \|\mathbf{S}^{\boldsymbol{\varepsilon}}_{0\perp}\| \lesssim \varepsilon^{3/2}$$

Error analysis

Conclusion

$$\|\hat{\boldsymbol{e}}_\perp\|_{H^{1/2}} \lesssim \left(\|\hat{\boldsymbol{g}}_\perp^{\boldsymbol{\varepsilon}}\| + \varepsilon^{1/2}\|\hat{\boldsymbol{g}}_0^{\boldsymbol{\varepsilon}}\|_{L^2}\right),$$

$$\|\hat{\boldsymbol{e}}_0\|_{L^2} \lesssim \varepsilon^{1/2}\|\hat{\boldsymbol{e}}_\perp\|_{H^{-1/2}}.$$

Bounds on the operators (following ideas of Hassan, Stamm '21)

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With constants depending on $W^{k,\infty}(\mathbb{R}^2)$-norms of u^{inc}:

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Take-away message

The convergence is assured by

$\|\hat{\boldsymbol{g}}_\perp^{\boldsymbol{\varepsilon}}\|_{H_*^{1/2}} \lesssim \varepsilon$ (behaviour of the data on the space orthogonal to the Galerkin space),

$\|\mathbf{S}_{\perp 0}^{\boldsymbol{\varepsilon}}\| = \|\mathbf{S}_{0\perp}^{\boldsymbol{\varepsilon}}\| \lesssim \varepsilon^{3/2}$ (off-diagonal operator terms)

Final result (frequency domain)

Summary

$\|\hat{\boldsymbol{e}}_0^{\boldsymbol{\varepsilon}}\|_{L^2} \lesssim \varepsilon^{3/2}$, $\|\hat{\boldsymbol{e}}_\perp^{\boldsymbol{\varepsilon}}\|_{H_*^{-1/2}} \lesssim \varepsilon$. Thus

$$\|\hat{\boldsymbol{\mu}}^{\boldsymbol{\varepsilon}} - \hat{\boldsymbol{\mu}}_{\boldsymbol{G}}^{\boldsymbol{\varepsilon}}\|_{H^{-1/2}(\Gamma^{\boldsymbol{\varepsilon}})} = \left(\|\hat{\boldsymbol{e}}_0^{\boldsymbol{\varepsilon}}\|_{L^2}^2 + \|\hat{\boldsymbol{e}}_\perp^{\boldsymbol{\varepsilon}}\|_{H_*^{-1/2}}^2\right)^{1/2} \lesssim \varepsilon$$

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$$\hat{u}^{\boldsymbol{\varepsilon}}(\boldsymbol{x}) - \hat{u}_G^{\boldsymbol{\varepsilon}}(\boldsymbol{x}) = \int_{\Gamma^{\boldsymbol{\varepsilon}}} G_\omega(\boldsymbol{x}-\boldsymbol{y})\,(\hat{\boldsymbol{\mu}}^{\boldsymbol{\varepsilon}} - \hat{\boldsymbol{\mu}}_G^{\boldsymbol{\varepsilon}})\,d\Gamma_{\boldsymbol{y}} = \hat{\nu}_0^{\boldsymbol{\varepsilon}}(\boldsymbol{x}) + \hat{\nu}_\perp^{\boldsymbol{\varepsilon}}(\boldsymbol{x}),$$

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Final result (time-domain, simplified)

Convergence of the density

$$\|\boldsymbol{\mu}^{\varepsilon} - \boldsymbol{\mu}_G^{\varepsilon}\|_{L^\infty(0,T;H^{-1/2}(\Gamma^{\varepsilon}))} \leq \varepsilon \times C_\mu \|u^{inc}\|_{H^8(0,T;W^{1,\infty}(\mathbb{R}^2))}$$

The constant C_μ depends polynomially on (the smallest distance between particles)$^{-1}$, number of particles, final time.

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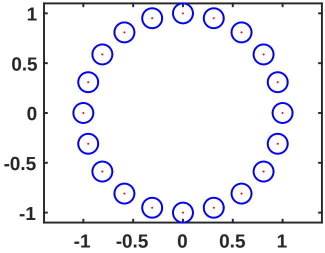
Super-convergence of the solution

Let $\boldsymbol{x} \in \Omega^{1,c}$. We have, as $\varepsilon \to 0$,

$$\|u^{\varepsilon}(., \boldsymbol{x}) - u_G^{\varepsilon}(., \boldsymbol{x})\|_{L^\infty(0,T)} \leq \varepsilon^2 \times C_u \|u^{inc}\|_{H^8(0,T;W^{1,\infty}(\mathbb{R}^2))}.$$

Numerical experiments



Data

$r_i^1 = r = 0.1$, data: $u^{inc}(t, \boldsymbol{x}) = \mathrm{e}^{-100(t-\boldsymbol{d}\cdot\boldsymbol{x}-2)^2}$
($\lambda_{min} \approx 0.1$)

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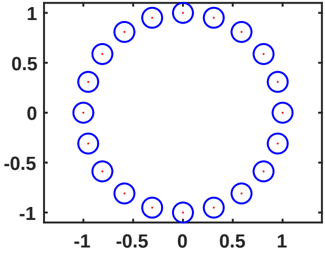
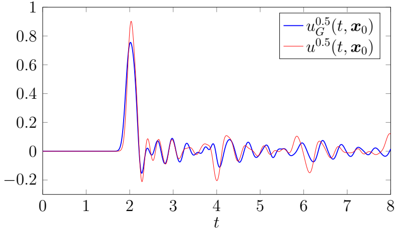
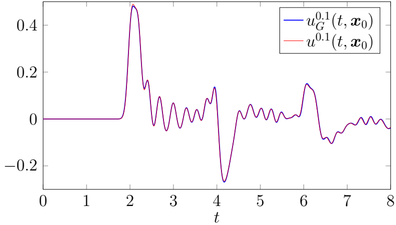
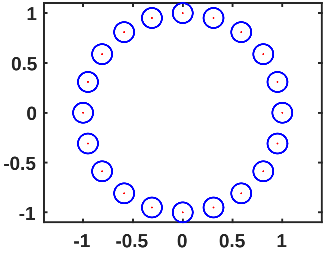
Experiment: compute $u^{\varepsilon}(t, \boldsymbol{x}_0)$, $\boldsymbol{x}_0 = 0$ depending on t









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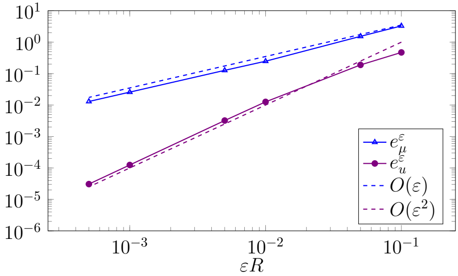
Experiment: convergence studies



MK, A new class of uniformly stable time-domain Foldy-Lax models for scattering by small particles. Acoustic sound-soft scattering by circles. To appear in SIAM:MMS

Other particles

This is a joint work with A. Savchuk (PhD student)

What happens to other particles

Same setting (sound-soft scattering), but with circles replaced by arbitrary Lipschitz domains?

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Find $\boldsymbol{\mu}_G^{\varepsilon} \in C^{\ell}(0, T; \mathbb{R}^N)$, s.t.

$$\int_{\Gamma_k^{\varepsilon}} g_k^{\varepsilon}(t, \mathbf{x}) d\Gamma_x = \sum_{n=1}^{N} \int_0^t \left(\int_{\Gamma_k^{\varepsilon}} \int_{\Gamma_n^{\varepsilon}} \mathcal{G}(t-\tau, \|\mathbf{x}-\mathbf{y}\|) d\Gamma_y \, d\Gamma_x \right) \mu_{G,n}^{\varepsilon}(\tau) d\tau.$$

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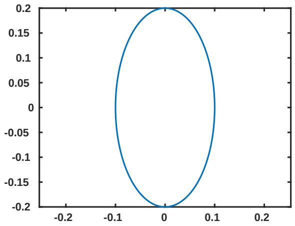
The first idea

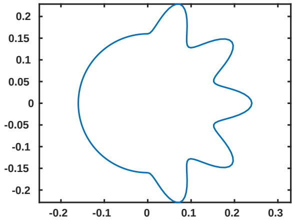
The Galerkin FL model as above can be defined for any shape, and is a priori stable. Let's see whether it converges.

Numerics

Data

The source $u^{inc}(t, \boldsymbol{x}) = \mathrm{e}^{-20(t-\boldsymbol{d}\cdot\boldsymbol{x}-2)^2}$, measure the error on $t \in (0, 4)$ at $\boldsymbol{x} = (0.2, 0.2)$





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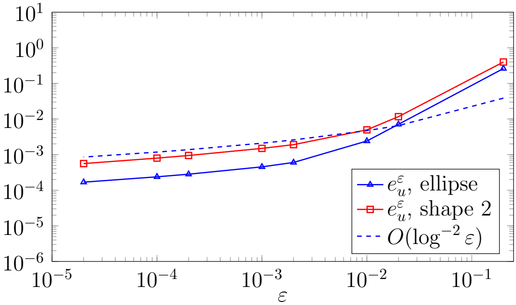

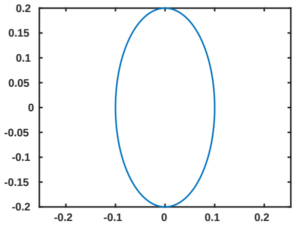

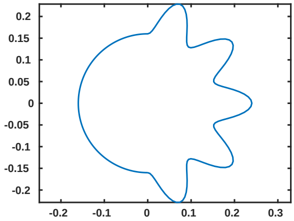

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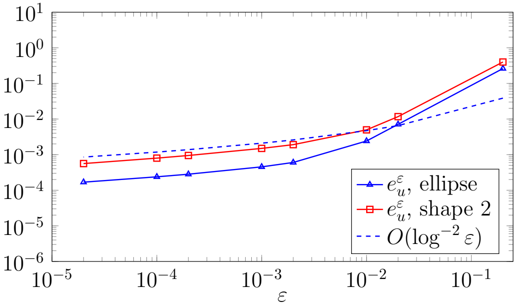

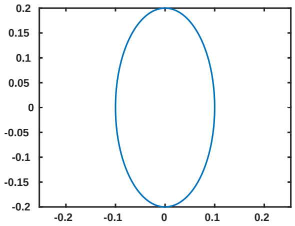

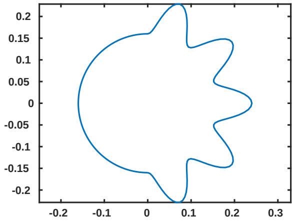

We have convergence in the relative error $O(\log^{-1}\varepsilon)$ (a 2D artifact, in 3D no convergence), but we would like to have higher order. Open question: the error is still not bad!

The outline of what follows

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Why the model does not converge well (where the circular shape comes into play)

Simplification: frequency domain, one $C^{2,\alpha}$ particle (centered in 0).
Goal: Find a good approximation to the exact solution and compare it with the one obtained through the Foldy-Lax model

The Galerkin Foldy-Lax model

$$\hat{\mu}_G^{\varepsilon} \iint_{\Gamma^{\varepsilon} \times \Gamma^{\varepsilon}} G_\omega(\|\boldsymbol{x} - \boldsymbol{y}\|) d\Gamma_{\boldsymbol{y}} d\Gamma_{\boldsymbol{x}} = -\int_{\Gamma^{\varepsilon}} \hat{u}^{inc}(\boldsymbol{x}) d\Gamma_{\boldsymbol{x}}, \quad G_\omega(r) = \frac{i}{4} H_0^{(1)}(\omega r).$$

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$$\int_{\Gamma^{\varepsilon}} G_\omega(\|\boldsymbol{x} - \boldsymbol{y}\|) \hat{\mu}^{\varepsilon}(\boldsymbol{y}) d\Gamma_{\boldsymbol{y}} = -\hat{u}^{inc}(\boldsymbol{x}) = -\hat{u}^{inc}(0) + O(\varepsilon), \quad \boldsymbol{x} \in \Gamma^{\varepsilon}.$$

Why the model does not converge well (where the circular shape comes into play)

Simplification: frequency domain, one $C^{2,\alpha}$ particle (centered in 0).
Goal: Find a good approximation to the exact solution and compare it with the one obtained through the Foldy-Lax model

The Galerkin Foldy-Lax model

$$\hat{\mu}_G^\varepsilon \iint_{\Gamma^\varepsilon \times \Gamma^\varepsilon} G_\omega(\|\boldsymbol{x}-\boldsymbol{y}\|)d\Gamma_{\boldsymbol{y}} d\Gamma_{\boldsymbol{x}} = -\int_{\Gamma^\varepsilon} \hat{u}^{inc}(\boldsymbol{x})d\Gamma_{\boldsymbol{x}}, \quad G_\omega(r) = \frac{i}{4}H_0^{(1)}(\omega r).$$

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Rescaling

With $\boldsymbol{y} = \varepsilon\hat{\boldsymbol{y}}$, $\boldsymbol{x} = \varepsilon\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}} \in \Gamma^1$,

$$\varepsilon \int_{\Gamma^1} G_\omega(\varepsilon\|\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}}\|)\hat{\mu}^\varepsilon(\varepsilon\hat{\boldsymbol{y}})d\Gamma_{\hat{\boldsymbol{y}}} = -\hat{u}^{inc}(0) + O(\varepsilon), \quad \hat{\boldsymbol{x}} \in \Gamma^1.$$

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Approximation

$$H_0^{(1)}(\varepsilon\omega\|\mathbf{x}-\mathbf{y}\|) = \frac{2i}{\pi}\log(\varepsilon\omega\|\mathbf{x}-\mathbf{y}\|) + C + O(\varepsilon\log\varepsilon), \quad \varepsilon\to 0+,$$

and this induces the decomposition

$$\mathbf{S}^1_{\omega\varepsilon} = \mathbf{S}^1_0 + \underbrace{\left(C - \frac{1}{2\pi}\log\varepsilon\omega\right)}_{C_{\omega\varepsilon}} I_{\Gamma^1} + o(1),$$

$$\mathbf{S}^1_0\varphi = -\frac{1}{2\pi}\int_{\Gamma^1}\log\|\mathbf{x}-\mathbf{y}\|\varphi(\mathbf{y})d\Gamma_{\mathbf{y}}, \quad I_{\Gamma^1}\varphi = \int_{\Gamma^1}\varphi d\Gamma.$$

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$$\mathbf{S}_0^1\varphi = -\frac{1}{2\pi}\int_{\Gamma^1} \log\|\boldsymbol{x}-\boldsymbol{y}\|\varphi(\boldsymbol{y})d\Gamma_{\boldsymbol{y}}, \quad I_{\Gamma^1}\varphi = \int_{\Gamma^1}\varphi d\Gamma.$$

$$\left(\mathbf{S}_0^1 + C_{\omega\varepsilon}I_{\Gamma^1} + o(1)\right)\hat{\mu}^{\varepsilon}(\varepsilon\cdot) = -\varepsilon^{-1}\hat{u}^{inc}(0) + O(1), \qquad \boldsymbol{x}\in\Gamma^1$$

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i.e. $\mathbf{S}_0^1\hat{\mu}^{\varepsilon}(\varepsilon\cdot) \approx \mathrm{const}(\omega, \varepsilon, \hat{u}^{inc}(0))$

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Final result

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The error: $\hat{e}^{\varepsilon} = \hat{e}_0^{\varepsilon} + \hat{e}_\perp^{\varepsilon}$, and $\hat{e}_\perp^{\varepsilon} = \hat{\mu}_\perp^{\varepsilon} \approx C_\mu\varepsilon^{-1}\log^{-1}\varepsilon\times\sigma_\perp(\varepsilon^{-1}\cdot) + o(\varepsilon^{-1}\log^{-1}\varepsilon)$

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(the above is sharp, see Reichel '97: for **non-circular domains**, $\sigma_\perp \neq 0$)

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Constant component: by a direct computation (with Galerkin Orthogonality):

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$$\|\hat{\mu}_G^{\varepsilon} - \hat{\mu}_0^{\varepsilon}\|_{L^2(\Gamma^{\varepsilon})} = O(\varepsilon^{-1/2}\log^{-2}\varepsilon) \quad O(\varepsilon^{3/2}) \text{ for many circles, 0 for 1 circle}$$

An alternative idea

Inspired by Challa, Sini 2013, Sini, Wang, Yao 2021

Exact density

$$\hat{\mu}^{\varepsilon}(\boldsymbol{y}) = \hat{u}^{inc}(0) \times \frac{\varepsilon^{-1}}{c_1 + c_2 \log \varepsilon} \sigma(\varepsilon^{-1}\boldsymbol{y}) + o(\varepsilon^{-1} \log^{-1} \varepsilon)_{L^\infty}, \quad \boldsymbol{y} \in \Gamma^{\varepsilon},$$

where $\sigma \in L^2(\Gamma^1)$ is a unique solution to $-\frac{1}{2\pi} \int_{\Gamma^1} \log \|\boldsymbol{x} - \boldsymbol{y}\| \sigma(\boldsymbol{y}) d\Gamma_{\boldsymbol{y}} = 1$.

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Galerkin space

Let σ_k be given by

$$\int_{\Gamma_k^1} G_0(\boldsymbol{x}, \boldsymbol{y}) \sigma_k(\boldsymbol{y}) = 1, \ \boldsymbol{x} \in \Gamma_k^1, \quad G_0(\|\boldsymbol{x} - \boldsymbol{y}\|) = -\frac{1}{2\pi} \log \|\boldsymbol{x} - \boldsymbol{y}\|.$$

The Galerkin space is then $\mathcal{V}_G^{\varepsilon} := \prod_{k=1}^{N} \operatorname{span}\{\sigma_k(\varepsilon^{-1}\boldsymbol{y})\}$.

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The Galerkin Foldy-Lax model

Find $\boldsymbol{\mu}_G^{\varepsilon} \in C^{\ell}(0, T; \mathbb{R}^N)$, s.t.

$$\int_{\Gamma_k^{\varepsilon}} g_k^{\varepsilon}(t, \mathbf{x}) \sigma_k(\varepsilon \mathbf{x}) d\Gamma_x = \sum_{n=1}^{N} \int_0^t \left(\iint_{\Gamma_k^{\varepsilon} \times \Gamma_n^{\varepsilon}} \mathcal{G}(t - \tau, \|\mathbf{x} - \mathbf{y}\|) \sigma_k(\varepsilon \mathbf{x}) \sigma_n(\varepsilon \mathbf{y}) d\Gamma_x \, d\Gamma_y \right) \mu_{G,n}^{\varepsilon}(\tau) d\tau.$$

A brief summary

- for circles, we construct the Galerkin Foldy-Lax model by taking $\mathcal{V}_G^{\varepsilon} = \mathbb{S}_0^{\varepsilon}$ (constants)

[1] And these are only $C^{2,\alpha}$-domains for which this is the case

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What follows: outline of the error analysis for arbitrary Lipschitz domains ($O(\varepsilon^2)$ convergence)

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Error analysis

Remark: we do the analysis in the frequency domain, like before

Problem

Before we had

$$H^s(\Gamma^{\varepsilon}) = \overbrace{\mathbb{S}_0^{\varepsilon}}^{\text{consts}} + \overbrace{H_*^s(\Gamma^{\varepsilon})}^{\perp \text{to consts}}, \quad s \in \{-1/2, 0, 1/2\},$$

the decomposition being orthogonal for $s \in \{0, 1/2\}$.
The convergence was achieved due to $\|\mathbb{P}_{\perp} \mathbf{S}_{\omega}^{\varepsilon} \mathbb{P}_0{}^*\| = O(\varepsilon^{3/2})$ (not true in our case)

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$$H^{-1/2}(\Gamma^\varepsilon) = \mathcal{V}_G^\varepsilon \dot{+} H_*^{-1/2}(\Gamma^\varepsilon).$$

(holds because $\sigma_k(\varepsilon^{-1}.) \notin H_*^{-1/2}(\Gamma_k^\varepsilon)$)

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This decomposition is not **orthogonal**, but direct. We have, for any $v \in H^{-1/2}(\Gamma^{\varepsilon})$:

$$v = v_{\sigma} + v_{\sigma,\perp}, \qquad v_{\sigma} \in \mathcal{V}_G^{\varepsilon}, \quad v_{\sigma,\perp} \in H_*^{-1/2}(\Gamma^{\varepsilon}).$$

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Projectors: $\mathbb{Q}_\sigma^* : v \mapsto v_\sigma$, $\mathbb{Q}_{\sigma,\perp}^* : v \mapsto v_{\sigma,\perp}$.

Some norm estimates, independent of ϵ

- $\|\mathbb{Q}_\sigma^*\| \lesssim 1$, $\|\mathbb{Q}_{\sigma,\perp}^*\| \lesssim 1$
- moreover, $\mathbb{P}_0^* \mathbb{Q}_\sigma^* = \mathbb{P}_0^*$

A discussion of the new decomposition

Before

The convergence was achieved due to $\|\mathbb{P}_\perp \mathbf{S}_\omega^{\varepsilon} \mathbb{P}_{\mathbf{0}}^*\| = O(\varepsilon^{3/2})$ (does not hold true for arbitrarily shaped particles).

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Our analysis follows the same lines as before, but uses both decompositions (the old and the new one). The crucial property, which is a counterpart of the above is

Now

$\|\mathbb{P}_\perp \mathbf{S}^{\varepsilon}_{\omega} \mathbb{Q}_{\sigma}^*\| \leq C(\omega)\varepsilon^{3/2}$ whenever $|\omega\varepsilon| < \text{const}$.

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only if $|\omega\varepsilon|$ is sufficiently small (thus 'orthogonal' to $H_*^{-1/2}$-space).

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Passage to the time-domain

Our estimates for the circles were uniform in frequency. Here we have convergence estimates for $|\omega\varepsilon| < 1/2$ only. How do we pass to the time domain?

Passage to the time domain

Main idea

Trade regularity for convergence by using the stability estimate

Passage to the time domain

Density error: low frequencies

$$\|\hat{\mathbf{e}}^{\varepsilon}\| \lesssim \varepsilon(1+|\omega|)^{m_e} \max(1, (\operatorname{Im}\omega)^{-n_e})\|\hat{u}^{inc}\|_{H^{\ell_e}}, \quad |\omega\varepsilon| < 1/2.$$

Passage to the time domain

Density error: low frequencies

$$\|\hat{\mathbf{e}}^{\varepsilon}\| \lesssim \varepsilon(1+|\omega|)^{m_e}\max(1,(\operatorname{Im}\omega)^{-n_e})\|\hat{u}^{inc}\|_{H^{\ell_e}}, \quad |\omega\varepsilon| < 1/2.$$

Density error: stability estimate

To obtain the error of the density for high frequencies, use the triangle inequality

$$\hat{\mathbf{e}}^{\varepsilon} = \hat{\boldsymbol{\mu}}^{\varepsilon} - \hat{\boldsymbol{\mu}}_G^{\varepsilon} \implies \|\hat{\mathbf{e}}^{\varepsilon}\| \leq \|\hat{\boldsymbol{\mu}}^{\varepsilon}\| + \|\hat{\boldsymbol{\mu}}_G^{\varepsilon}\|$$

and next a stability estimate (valid for any $\omega : \operatorname{Im}\omega > 0$)

$$\|\hat{\mathbf{e}}^{\varepsilon}\| \lesssim \varepsilon^{-1/2}(1+|\omega|)^{m_s}\max(1,(\operatorname{Im}\omega)^{-n_s})\|\hat{u}^{inc}\|_{H^{\ell_s}}$$

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Density error: high frequencies

For $|\omega\varepsilon| > 1/2$, we have $\varepsilon^{-1} < 2|\omega|$, thus

$$\begin{aligned}\|\hat{\mathbf{e}}^{\varepsilon}\| &\lesssim \varepsilon \times \varepsilon^{-1}\varepsilon^{-1/2}(1+|\omega|)^{m_s}\max(1,(\operatorname{Im}\omega)^{-n_s})\|\hat{u}^{inc}\|_{H^{\ell_s}} \\ &\lesssim \varepsilon \times |\omega|^{3/2}(1+|\omega|)^{m_s}\max(1,(\operatorname{Im}\omega)^{-n_s})\|\hat{u}^{inc}\|_{H^{\ell_s}}\end{aligned}$$

Passage to the time domain

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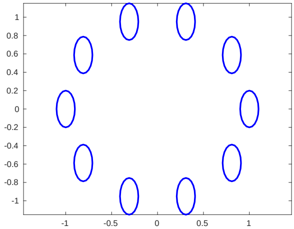
Back in the time-domain, the powers of ω turn into derivatives $\implies$ regularity

Numerical results

Data

Scattering by many ellipses, the source $u^{inc}(t, \boldsymbol{x}) = \mathrm{e}^{-20(t-\boldsymbol{d}\cdot\boldsymbol{x}-2)^2}$, measure the absolute error on $t \in (0, 8)$ at $\boldsymbol{x} = (0, 0)$



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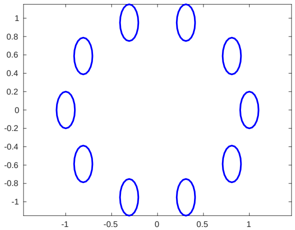

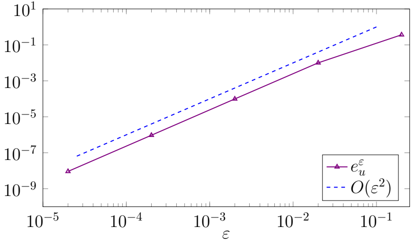

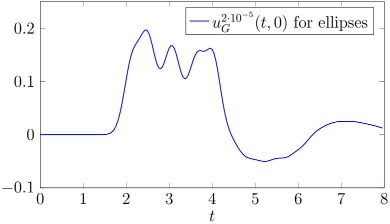

A list of open questions

- high-order methods for arbitrary particles
- particles close to each other
- 3D Maxwell
- dispersive problems