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Funded by
DFG Deutsche Forschungsgemeinschaft
German Research Foundation

Resonance-free regions and structural optimization of scattering poles.

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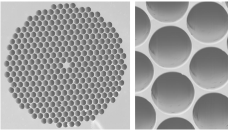
The talk is based on the paper (K., Koch, Verbytskyi '20)

with the background information from

(K. '11-14), (K., Logachova, Verbytskyi '17), (Albeverio, K. '17).

Spectral and Resonance Problems for Imaging, Seismology and Materials Science,
the University of Reims, 20-24.11.2023

Photonic crystal designs of high-Q optical cavities



SEM images of US NRL-produced photonic crystal fiber.
The diameter of the central solid core is 5 µm,
the diameter of the holes is 4 µm.

https://en.wikipedia.org/wiki/Photonic_crystal#/media/File:Photonic-crystal-fiber-from-NRL.jpg

$$\partial_t \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\epsilon_0 \varepsilon(x)} \nabla \times \\ -\frac{1}{\mu_0} \nabla \times & 0 \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}$$

$$\varepsilon(x) = \begin{cases} 1, & x \notin \Omega \\ \text{nonhomogeneous structure}, & x \in \Omega \end{cases}, \quad \Omega \subset \mathbb{R}^3 \text{ is bounded.}$$

Solutions $e^{-\mathrm{i}\omega t} \begin{pmatrix} \widetilde{\mathbf{E}}(x) \\ \widetilde{\mathbf{H}}(x) \end{pmatrix}$ with outgoing eigenmodes $\begin{pmatrix} \widetilde{\mathbf{E}}(x) \\ \widetilde{\mathbf{H}}(x) \end{pmatrix}$ essentially correspond to continuation resonances (scattering poles) ω.

$$\partial_t \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\epsilon_0 \varepsilon(x)} \nabla \times \\ -\frac{1}{\mu_0} \nabla \times & 0 \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \qquad \varepsilon(x) = 1 \text{ for } x \notin \Omega.$$

The set $\Sigma(\varepsilon)$ of resonances $\omega \in \overline{\mathbb{C}}_- = \{\operatorname{Im} z \leqslant 0\}$ is understood as the set of generalized eigenvalues associated with the radiation condition at ∞ (Sommerfeld/Silver-Müller radiation condition),

or

as the set of poles of the resolvent $(M_\varepsilon - \omega)^{-1}$ with a spatial cut-off analytically continued through the essential spectrum.

$\alpha = \operatorname{Re} \omega$ is the (real-)frequency of eigen-oscillations $e^{-\mathrm{i}\omega t} \begin{pmatrix} \widetilde{\mathbf{E}}(x) \\ \widetilde{\mathbf{H}}(x) \end{pmatrix}$,

$\beta = \operatorname{Dr}(\omega) = -\operatorname{Im} \omega \geqslant 0$ is the decay rate.

Studies of photonic crystals having ω with high quality-factor

$$Q(\omega) = -\frac{1}{2}\frac{\mathrm{Re}\,\omega}{\mathrm{Im}\,\omega}$$

were inspired by miniaturization of Schrödinger 'cat' experiments.
Cavity QED experiments, 2012 Nobel Prize of Serge Haroche and David Wineland.

Some of the designs:.

- Fabrication of 2-D photonic crystals with high-Q (Akahane, Asano, Song, Noda '03):

  "... light should be confined gently in order to be confined strongly."

  Design (A): centered defect in a periodic structure.

- Numerically simulated 1-D photonic crystals with high-Q (Notomi, Kuramochi, Taniyama '08),
  fabrication (Kuramochi,Taniyama, Tanabe, Kawasaki,Roh, Notomi '10) :

  Design (B): gradually changed deviations from periodically alternating layers.

Some of the methods.

Numerical local maximization over a finite set of structural parameters (Liang, Johnson '13).

- (Asano, Noda '19) machine learning, deep neural network.
- (Vasco, Savona '21) particle swarm algorithms on multi-core architecture, modelling of random imperfections.
- (Fukuda, Asano, Takahashi, Noda '22) non-symmetric high-Q cavities.

Constraints:

- $\varepsilon(x) = \epsilon_1 = 1$ or $\varepsilon(x) = \epsilon_2$ for x in bounded $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, the feasible family (set)
  $\mathbb{F}_0 = \{\varepsilon \in L^\infty(\Omega) \ : \ \varepsilon(x) = \epsilon_1 \chi_{\Omega_1}(x) + \epsilon_2 \chi_{\Omega_2}(x), \quad \Omega_2 = \Omega \backslash \Omega_1\}$
- For $d = 1$, the relaxed family
  $\mathbb{F} = \{\varepsilon \in L^\infty(\Omega) \ : \ \epsilon_1 \leqslant \varepsilon(x) \leqslant \epsilon_2 \quad \text{a.e.}\}$.

Some of the studies with the numerical minimization of the decay rate:

- (Kao, Santosa '08) 1-D and 2-D,
  search of local minimizers of $|\operatorname{Im}\omega|$ by iterative steepest ascent method.
  Difficulty with multiple resonances in 2-D is explicitly noticed.
- (Heider, Berebichez, Kohn, Weinstein '08) 1-D, steepest ascent method.
- (Osting, Weinstein '13), a nonexistence conjecture based on numerical evidence:
  $$\operatorname*{arg\,min}_{\substack{\varepsilon\in\mathbb{F}\\ \omega\in\Sigma(\varepsilon)}} \mathrm{Dr}(\omega) = \varnothing, \qquad \inf_{\substack{\varepsilon\in\mathbb{F}\\ \omega\in\Sigma(\varepsilon)}} \mathrm{Dr}(\omega) = 0.$$
- (Ogasawara '14, Bachelor thesis, UBC)
  under supervision of Richard Froese,
  Matlab's built-in optimization, 1-D Schrödinger eq. with δ-interactions,
  explicit nonexistence conjecture.

Difficulty: sliding of iterations ω_n to ∞ $\rightarrow$ (non)existence of optimizers.

Observations: high dielectric contrast designs,
close to periodic patterns with a defect in the center
(Kao, Santosa '08), (Heider, Berebichez, Kohn, Weinstein '08).

The pioneering paper for 1-D and 3-D Schrödinger eq-s (Harrell, Svirsky '86) .

Motivation: estimation of resonances of random potentials.
The high contrast theorem for optimal designs under
the additional assumption that optimal resonance is simple.

Main difficulties are identified:
multiple resonances, (non)existence of optimizers.

Multiple resonaces may exist even for 1-D Schrödinger operator (Korotyaev '04), examples for Krein strings with δ-masses (van den Brink, Young '01), (K. '13).

Analytic minimization of the decay rate $\mathrm{Dr}(\omega) = |\operatorname{Im} \omega|$.

Pareto optimization formulation for 1-D optical cavities (TEM-waves):

(K. '11-13), (K., Logachova, Verbytskyi '17). Rigorous existence of optimizers.

(K. '14) Krein strings with the total mass constraints, hyperbolic billiard.

The analytic method of multi-parameter perturbations of resonances, including multiple resonances.

Special examples of explicitly calculated Pareto minimizers:

(K. '11-13) special Krein-Nudelman strings, the trace-type formulae method;

(K. '14) low frequency region under the total mass constraints on the string;

(Albeverio, K. '17) point interactions in 3-D,
symmetry breaking, nonuniqueness of optimizers.

The optimal control approach to 1-D optical cavities:

(K., Koch, Verbytskyi '20) reformulation of Pareto optimization of resonances as an optimal control problem (partially equivalent).

Maximum Principle, Hamilton-Jacobi-Bellman (HJB-) equation, extremal synthesis.

For symmetric cavities, combination of an analytically derived nonlinear eigenproblem with a special numerical shooting method allowed us to compute optimal structures.

Analytic background: resonances in 1-D photonic crystals for TEM waves

The set $\Sigma(\varepsilon)$ of resonances ω of consists of eigenvalues of

$$-y''(s) = \omega^2 \varepsilon(s) y(s)$$
$$-\mathrm{i}\frac{y'(s_\pm)}{\omega} = \pm\sqrt{\epsilon_\infty} y(s_\pm) \qquad \text{(radiation boundary conditions).}$$

$0 < \epsilon_1 \leqslant \varepsilon(s) \leqslant \epsilon_2$ for $s_- < s < s_+$ is the structure of a resonator,

$\varepsilon(s) \equiv \epsilon_\infty > 0$ for $s \notin [s_-, s_+]$ is the homogeneous outer medium.

$\Sigma(\varepsilon) = \{\omega_j\} \subset \mathbb{C}_- := \{\operatorname{Im} z < 0\}$ is symmetric w.r.t. $\mathrm{i}\mathbb{R}$.

(a) non-constant $\varepsilon(\cdot) : [s_-, s_+] \to \mathbb{R}_+$ (b) $\varepsilon(\cdot) \equiv \text{const} \neq \epsilon_\infty$, $s \in [s_-, s_+]$

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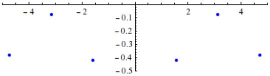

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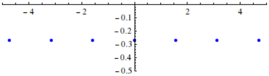

(b) $\varepsilon(\cdot) \equiv \mathrm{const} \neq \epsilon_\infty$, $s \in [s_-, s_+]$

Resonances in symmetric 1-D photonic crystals

Let $0 < \epsilon_1 \leqslant \varepsilon(s) \leqslant \epsilon_2$ for $-\ell < s < \ell$ describe the resonator.
Let $\varepsilon(s) \equiv \epsilon_\infty > 0$ for $s \notin [-\ell, \ell]$ be the homogeneous outer medium.

Assume additionally that ε is even, $\varepsilon(s) = \varepsilon(-s)$ (symmetry w.r.t. $s = 0$).

$$-y''(s) = \omega^2 \varepsilon(s) y(s), \qquad -\mathrm{i}\frac{y'(\pm\ell)}{\omega} = \pm\sqrt{\epsilon_\infty}\, y(\pm\ell),$$

$$\text{additionally} \qquad y'(0) = 0, \qquad \text{or} \qquad y(0) = 0.$$

Resonance modes y are either even, or odd.

$\Rightarrow$ Reduction to $s \in [0, \ell]$ (or equivalently to $s \in [-\ell, 0]$).

The set $\Sigma(\varepsilon) = \Sigma^{\text{even}}(\varepsilon) \cup \Sigma^{\text{odd}}(\varepsilon)$ is the disjoint union of the sets of even-mode and odd-mode resonances.

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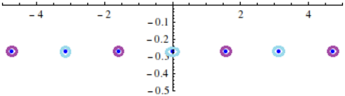

The case $\varepsilon \equiv \text{const} > \epsilon_\infty$.

Pareto optimization and the resonance-free region.

$$-y''(s) = \omega^2 \varepsilon(s) y(s), \qquad -\frac{\mathrm{i}y'(s_\pm)}{\omega} = \pm\sqrt{\epsilon_\infty}\, y(s_\pm)$$

Let us fix the resonator region $[s_-, s_+]$ and the outer permittivity ϵ_∞.

The constraints $0 < \epsilon_1 \leqslant \varepsilon(s) \leqslant \epsilon_2$ define the feasible family:
$\mathbb{F}_{s_-,s_+} := \{ \varepsilon(x) \in L^\infty_{\mathbb{R}}(s_-, s_+) \; : \quad \epsilon_1 \leqslant \varepsilon(x) \leqslant \epsilon_2 \quad \text{a.e.}\}$.

The set of achievable resonances is $\Sigma[\mathbb{F}_{s_-,s_+}] := \bigcup_{\varepsilon \in \mathbb{F}_{s_-,s_+}} \Sigma(\varepsilon)$.

The set $\quad \mathbb{C}\setminus\Sigma[\mathbb{F}_{s_-,s_+}] \quad$ is the resonance-free region (over $\mathbb{F}_{s_-,s_+}$).

Main idea: "Pareto extremal" resonances are on the boundary $\partial\Sigma[\mathbb{F}_{s_-,s_+}]$,
Pareto optimal resonances is the part of the boundary "closer to the real line".

Existence theorem for optimizers:
the set of achievable resonances $\Sigma[\mathbb{F}_{s_-,s_+}]$ is closed.

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The statement of the Pareto optimization problem for $\mathrm{Dr}(\omega)$ (K.'11-14)

Problem 1. Pareto optimization of $\mathrm{Dr}(\omega) = -\operatorname{Im}\omega$.

$-y''(s) = \omega^2 \varepsilon(s) y(s), \qquad -\mathrm{i}y'(s_\pm)/\omega = \pm\sqrt{\epsilon_\infty}\, y(s_\pm)$

$\omega = \alpha - \mathrm{i}\beta$ with $\beta = \mathrm{Dr}(\omega) > 0$, while $\alpha \in \mathbb{R}$ is the frequency of a resonance ω.

The set of achievable frequencies $\operatorname{Re}\Sigma[\mathbb{F}_{s_-,s_+}] := \bigcup_{\omega\in\Sigma[\mathbb{F}_{s_-,s_+}]} \operatorname{Re}\omega$.

Def. (minimal decay rate $\beta_{\min}(\alpha)$ for an achievable frequency)

$$\beta_{\min}(\alpha) = \min_{\substack{\operatorname{Re}\omega=\alpha \\ \omega\in\Sigma[\mathbb{F}_{s_-,s_+}]}} |\operatorname{Im}\omega|$$

Def. (Pareto optimizers for particular achievable frequencies α)

$\omega_\alpha = \alpha - \mathrm{i}\beta_{\min}(\alpha)$ is the resonance of minimal decay (for achievable α).

If $\omega_\alpha \in \Sigma(\varepsilon)$ for $\varepsilon \in \mathbb{F}_{s_-,s_+}$, we say that ω_α and ϵ are of minimal decay (for α).

The Pareto (optimal) frontier is $\mathrm{Pa}_{\mathrm{Dr}} := \{\alpha - \mathrm{i}\beta_{\min}(\alpha) \;:\; \alpha \in \operatorname{Re}\Sigma[\mathbb{F}_{s_-,s_+}]\}$.

Theorem. For every achievable α, $\exists$ a structure ε of minimal decay.

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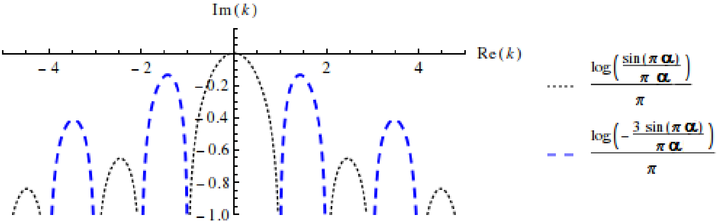
Theorem. For every achievable α, ∃ a structure ε of minimal decay.

Explicitly calculated Pareto frontiers for 3-D Schrödinger eq. (Albeverio, K.'17).

Pareto optimization of resonances for 3-D Schrödinger eq. with δ-interactions at vertices of a regular tetrahedron with varying "strength" parameters.



Resonances $\omega = \alpha - i\beta_{\min}(\alpha)$ of minimal decay in the equidistant case $L = \pi$, $N = 4$:

- - - - marks the case where minimal number of δ-interactions is $n_{\min}(\omega) = 4$,

$\cdots\cdots$ marks the case $n_{\min}(\omega) = 2$.

Weak symmetry breaking:

1) $\exists \alpha \in \mathbb{R}$ s.t. some of corresponding optimal structures do not possess all the symmetries of a regular tetrahedron.

2) For each $\alpha \in \mathbb{R}\backslash\{0\}$ $\exists$ exactly one optimal structure that possesses all the symmetries.

3) Each optimal structure possesses at least one of the symmetries.

Analytic background of Pareto optimization, the symmetry constraint.

$$-y''(s) = \omega^2 \varepsilon(s) y(s), \qquad -\mathrm{i}y'(\pm\ell)/\omega = \pm\sqrt{\epsilon_\infty}\, y(\pm\ell)$$

$$\mathbb{F}_\ell^{\mathrm{sym}} := \{ \varepsilon(s) \in L^\infty_{\mathbb{R}}(-\ell,\ell) \ : \ \epsilon_1 \leqslant \ \varepsilon(s) = \varepsilon(-s) \ \leqslant \epsilon_2 \text{ a.e. } \}.$$

The closed sets of odd-mode and even-mode achievable resonances,

$\Sigma^{\mathrm{odd}}[\mathbb{F}_\ell^{\mathrm{sym}}] := \bigcup_{\varepsilon \in \mathbb{F}_\ell^{\mathrm{sym}}} \Sigma^{\mathrm{odd}}(\varepsilon)$ and $\Sigma^{\mathrm{even}}[\mathbb{F}_\ell^{\mathrm{sym}}] := \bigcup_{\varepsilon \in \mathbb{F}_\ell^{\mathrm{sym}}} \Sigma^{\mathrm{even}}(\varepsilon)$.

The odd-mode and even-mode minimal decay rates are

$$\beta_{\min}^{\mathrm{odd}}(\alpha) = \min_{\substack{\mathrm{Re}\,\omega=\alpha \\ \omega\in\Sigma^{\mathrm{odd}}[\mathbb{F}_\ell^{\mathrm{sym}}]}} |\,\mathrm{Im}\,\omega|\,, \qquad \beta_{\min}^{\mathrm{even}}(\alpha) = \min_{\substack{\mathrm{Re}\,\omega=\alpha \\ \omega\in\Sigma^{\mathrm{even}}[\mathbb{F}_\ell^{\mathrm{sym}}]}} |\,\mathrm{Im}\,\omega|\,;$$

the Pareto frontiers: $\mathrm{Pa}_{\mathrm{Dr}}^{\mathrm{odd(even)}} := \{\alpha - \mathrm{i}\beta_{\min}^{\mathrm{odd(even)}}(\alpha) : \alpha \in \mathrm{Re}\,\Sigma^{\mathrm{odd(even)}}[\mathbb{F}_\ell^{\mathrm{sym}}]\}$.

$\Sigma^{\mathrm{odd}}[\mathbb{F}_\ell^{\mathrm{sym}}]$ (i.e., $y(0)=0$) and $\mathrm{Pa}_{\mathrm{Dr}}^{\mathrm{odd}}$ in the domain $\mathcal{D}$;
$\epsilon_1 = 90$, $\epsilon_2 = 110$ (low contrast), $\epsilon_\infty = 1$;
drawing based on Euler-Lagrange bang-bang eigenpr.
+ shooting meth. in $[-1,0]$ (K., Logachova, Verbytskyi '17).

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$\Sigma^{\mathrm{odd}}[\mathbb{F}_\ell^{\mathrm{sym}}] := \bigcup_{\varepsilon\in\mathbb{F}_\ell^{\mathrm{sym}}} \Sigma^{\mathrm{odd}}(\varepsilon)$ and $\Sigma^{\mathrm{even}}[\mathbb{F}_\ell^{\mathrm{sym}}] := \bigcup_{\varepsilon\in\mathbb{F}_\ell^{\mathrm{sym}}} \Sigma^{\mathrm{even}}(\varepsilon)$.

The odd-mode and even-mode minimal decay rates are

$$\beta^{\mathrm{odd}}_{\min}(\alpha) = \min_{\substack{\mathrm{Re}\,\omega=\alpha \\ \omega\in\Sigma^{\mathrm{odd}}[\mathbb{F}_\ell^{\mathrm{sym}}]}} |\,\mathrm{Im}\,\omega|\,, \qquad \beta^{\mathrm{even}}_{\min}(\alpha) = \min_{\substack{\mathrm{Re}\,\omega=\alpha \\ \omega\in\Sigma^{\mathrm{even}}[\mathbb{F}_\ell^{\mathrm{sym}}]}} |\,\mathrm{Im}\,\omega|\,;$$

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$\Sigma^{\mathrm{odd}}[\mathbb{F}_\ell^{\mathrm{sym}}]$ (i.e., $y(0) = 0$) and $\mathrm{Pa}^{\mathrm{odd}}_{\mathrm{Dr}}$ in the domain $\mathcal{D}$;
$\epsilon_1 = 90$, $\epsilon_2 = 110$ (low contrast), $\epsilon_\infty = 1$;
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Analytic background of Pareto optimization, the symmetry constraint.

$$-y''(s) = \omega^2 \varepsilon(s) y(s), \qquad -\mathrm{i}y'(\pm\ell)/\omega = \pm\sqrt{\epsilon_\infty}\, y(\pm\ell)$$

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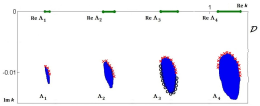

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The analogue of Euler-Lagrange equation for Pareto optimal eigenmodes y:

Theorem (K. '12-'13). Let $\omega = \alpha - i\beta_{\min}(\alpha)$ and $\omega \in \Sigma(\varepsilon)$ for $\varepsilon \in \mathbb{F}_{s_-,s_+}$. Then:
∃ optimal eigenmode $y \in W^{2,\infty}_{\mathbb{C}}[s_-, s_+]$ of $\varepsilon(\cdot)$ s.t.

$$-y'' = \omega^2 y[\epsilon_1 + (\epsilon_2 - \epsilon_1)\chi_{\mathbb{C}_+}(y^2)],$$
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and additionally $\varepsilon = \epsilon_1 + (\epsilon_2 - \epsilon_1)\chi_{\mathbb{C}_+}(y^2)$.

Here $\chi_{\mathbb{C}_+}(\zeta) := \begin{cases} 1 & \text{if } \operatorname{Im}\zeta > 0, \\ 0 & \text{if } \operatorname{Im}\zeta \leqslant 0. \end{cases}$

Proof. The method of multi-parameter perturbations of resonances
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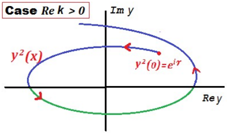

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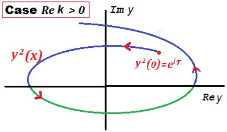

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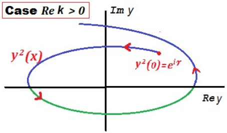

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Euler-Lagrange bang-bang eigenproblems and the shooting method on $[-\ell, 0]$.

Computation of nonlinear eigenvalues by the shooting

from $\mathrm{i}y'(-1)/\omega = \sqrt{\epsilon_\infty}y(-1)$ to $y(0) = 0$

$\epsilon_1 = 90$, $\epsilon_2 = 110$, $\epsilon_\infty = 1$

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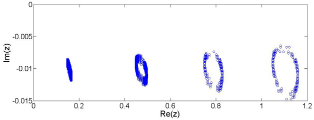

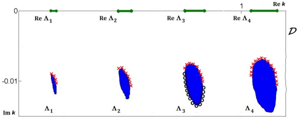

This produces the Pareto optimal frontier

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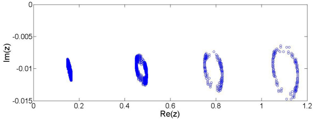

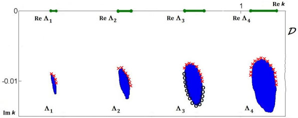

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Optimal control approach & minimum-time shooting method (K., Koch, Verbytskyi '20).

The general scheme of the method:

- An extremal form of Lagrange duality produces an optimal control (OC) problem on the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (the minimum-time problem).
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  The equivalence follows from scaling of Maxwell/string equations.
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Step 1. Minimization of resonator length for a fixed resonance $\omega \in \mathbb{C}_-$.

The family $\mathbb{F}$ consists of $\varepsilon \in L^\infty(\mathbb{R})$ s.t. $\mathfrak{n}_1^2 \leqslant \varepsilon(s) \leqslant \mathfrak{n}_2^2$ and

$$\text{there exist } s_\pm \text{ s.t. } \varepsilon(s) \equiv \mathfrak{n}_\infty^2 \text{ for } s \in \mathbb{R} \backslash [s_-, s_+], \quad (1)$$

here $\mathfrak{n}_j = (\epsilon_j)^{1/2}$ are refractive indices.

The effective length of the resonator ε is $L(\varepsilon) := s_+^\varepsilon - s_-^\varepsilon$,
where $[s_-^\varepsilon, s_+^\varepsilon]$ is the shortest interval satisfying (1)
(if $\varepsilon(\cdot) \equiv \mathfrak{n}_\infty^2$, we put $s_\pm^\varepsilon = 0$, and so $L(\varepsilon) = 0$).

Problem 2. $\displaystyle \arg\min_{\substack{\varepsilon \in \mathbb{F} \\ k \in \Sigma(\varepsilon)}} L(\varepsilon)$, we denote the minimal length $L_{\min}(\omega)$.

The set $\Sigma(\varepsilon)$ is well defined since for $-y''(s) = \omega^2 \varepsilon(s) y(s)$ and $\varepsilon \in \mathbb{F}$:

$-\mathrm{i}y'(s_\pm^\varepsilon)/\omega = \pm \mathfrak{n}_\infty y(s_\pm^\varepsilon)$ can be replaced by

$-\mathrm{i}y'(s_\pm)/\omega = \pm \mathfrak{n}_\infty y(s_\pm)$ with arbitrary $\pm s_\pm > \pm s_\pm^\varepsilon$ (radiation conditions).

The family $\mathbb{F}$ consists of $\varepsilon \in L^\infty(\mathbb{R})$ s.t. $\mathfrak{n}_1^2 \leqslant \varepsilon(s) \leqslant \mathfrak{n}_2^2$ and

$$\text{there exist } s_\pm \text{ s.t. } \varepsilon(s) \equiv \mathfrak{n}_\infty^2 \text{ for } s \in \mathbb{R} \setminus [s_-, s_+], \qquad (1)$$

here $\mathfrak{n}_j = (\epsilon_j)^{1/2}$ are refractive indices.

The effective length of the resonator ε is $L(\varepsilon) := s_+^\varepsilon - s_-^\varepsilon$,
where $[s_-^\varepsilon, s_+^\varepsilon]$ is the shortest interval satisfying (1)
(if $\varepsilon(\cdot) \equiv \mathfrak{n}_\infty^2$, we put $s_\pm^\varepsilon = 0$, and so $L(\varepsilon) = 0$).

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Step 1. Minimization of resonator length for a fixed resonance $\omega \in \mathbb{C}_-$.

Symmetric version: $\mathbb{F}^{\text{sym}} := \{\varepsilon \in \mathbb{F} \ : \ \varepsilon(s) = \varepsilon(-s)$ a.e. on $\mathbb{R}\}$,

$$\arg\min_{\substack{\varepsilon \in \mathbb{F}^{\text{sym}} \\ \omega \in \Sigma^{\text{odd}}(\varepsilon)}} L(\varepsilon), \qquad \arg\min_{\substack{\varepsilon \in \mathbb{F}^{\text{sym}} \\ \omega \in \Sigma^{\text{even}}(\varepsilon)}} L(\varepsilon),$$

the minimal lengths $L^{\text{odd}}_{\min}(\omega)$ and $L^{\text{even}}_{\min}(\omega)$.

Idea. Consider s as time. Find the minimal time $T_k^{\min}(-\mathfrak{n}_\infty, \mathfrak{n}_\infty) = s_+ - s_-$ needed to get from $\frac{y'(s_-)}{\mathrm{i}\omega y(s_-)} = -\mathfrak{n}_\infty$ (initial point)

to $\frac{y'(s_+)}{\mathrm{i}\omega y(s_+)} = \mathfrak{n}_\infty = \sqrt{\epsilon_\infty}$ (target).

The family of (feasible) controls $\mathbb{F}_{s_-} := \{\varepsilon \in L^\infty(s_-, +\infty) \ : \ \mathfrak{n}_1^2 \leqslant \varepsilon(s) \leqslant \mathfrak{n}_2^2\}$.

Using Riccati transform $x = \dfrac{y'}{\mathrm{i}\omega y}$, we rewrite $-y''(s) = \omega^2 \varepsilon(s) y(s)$ as the control system

$$x' = \mathrm{i}\omega(-x^2 + \varepsilon)$$

in $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Step 2. Minimum time control reformulation.

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Equivalence: $L_{\min}(\omega) = T_\omega^{\min}(-\mathfrak{n}_\infty, \mathfrak{n}_\infty)$,

$L_{\min}^{\text{even}}(\omega) = T_\omega^{\min}(-\mathfrak{n}_\infty, 0) = T_k^{\min}(0, \mathfrak{n}_\infty)$,

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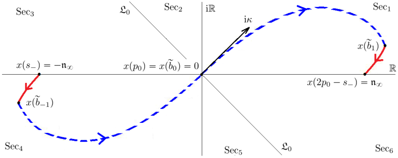
i.e., $x(s) = \infty$ corresponds to $y(s) = 0$.



Drawing based on a computed trajectory of the x-extremal and the optimizer $\varepsilon(\cdot)$ of the even-mode minimal length problem for $\omega = 1 - \mathrm{i}$, $\mathfrak{n}_1 = 1$, $\mathfrak{n}_2 = \mathfrak{n}_\infty = 3.46$.

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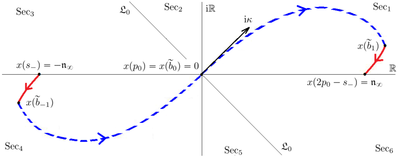

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Step 3. Pareto frontier of resonances of minimal $|\omega|$ for a given $\operatorname{Arg} k$.

Let $\Sigma^{s_-,s_+}_{\eta_-,\eta_+}(\varepsilon)$ be the set of eigenvalues of

$$-y''(s) = \omega^2 \varepsilon(s) y(s), \qquad \frac{y'(s_\mp)}{\mathrm{i}\omega y(s_\mp)} = \eta_\mp, \qquad \eta_\mp \in \widehat{\mathbb{C}}, \qquad \omega = \rho e^{\mathrm{i}\gamma}.$$

Problem 2'. Minimization of $|\omega|$ for a given achievable complex argument $\gamma = \operatorname{Arg}\omega \in (-\pi/2, 0)$ (the interval $[s_-, s_+]$ is fixed).

Achievable eigenvalues: $\Sigma^{s_-,s_+}_{\eta_-,\eta_+}[\mathbb{F}_{s_-}] := \bigcup_{\varepsilon \in \mathbb{F}_{s_-}} \Sigma^{s_-,s_+}_{\eta_-,\eta_+}(\varepsilon)$.

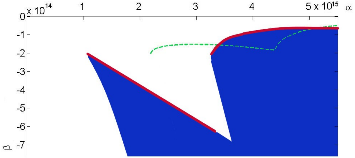
The minimal modulus for an argument $\gamma \in (-\pi/2, 0)$

$\rho_{\min}(\gamma) = \rho_{\min}(\gamma, \eta_-, \eta_+) := \inf\{|\omega| \ : \ \omega \in \Sigma^{s_-,s_+}_{\eta_-,\eta_+}[\mathbb{F}_{s_-}]$ and $\operatorname{Arg}\omega = \gamma\}$.

The Pareto frontier $\mathrm{Pa}^{\eta_-,\eta_+}_{\mathrm{mod}} := \{e^{\mathrm{i}\gamma}\rho_{\min}(\gamma) \ : \ \gamma \in \operatorname{Arg}\Sigma^{s_-,s_+}_{\eta_-,\eta_+}[\mathbb{F}_{s_-}]\}$.

Theorem (equivalence of Pr.2 and Pr.2').

$T^{\min}_{\omega}(\eta_-, \eta_+) = \frac{(s_+ - s_-)}{|\omega|}\rho_{\min}(\gamma, \eta_-, \eta_+)$.



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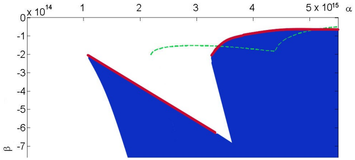

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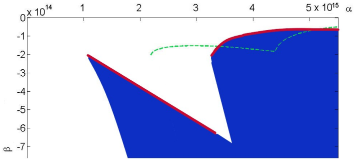

Proof. The natural scaling of eigenproblem:

if $\quad \omega \in \Sigma_{\eta_-,\eta_+}^{s_-,s_+}(\varepsilon) \quad$ and $\quad \widetilde{\varepsilon}(s) = \varepsilon(\tau s), \quad$ then $\quad \tau\omega \in \Sigma_{\eta_-,\ \eta_+}^{\tau^{-1}s_-,\ \tau^{-1}s_+}(\widetilde{\varepsilon})$.

The Hamilton-Jacobi-Bellman equation for $V(x) := T_\omega^{\min}(x, \mathfrak{n}_\infty)$:

$0 = 1 - \max\{-\nabla_{f(x,\epsilon)} V(x) \; : \; \epsilon = \epsilon_j, \; j = 1, 2\}$

where $\nabla_z V(x) := \lim\limits_{\substack{\zeta \to 0 \\ \zeta \in \mathbb{R}}} \frac{V(x+\zeta z) - V(x)}{\zeta}$ is the deriv. in the direct. $z \in \mathbb{C} = \mathbb{R}^2$.

The HJB equation can be written as the boundary value problem

$0 = V(\mathfrak{n}_\infty), \quad 0 = 1 - \nabla_{\mathrm{i}\omega(x^2 - (\epsilon_2 + \epsilon_1)/2)} V(x) - \frac{\epsilon_2 - \epsilon_1}{2} |\nabla_{\mathrm{i}\omega} V(x)|,$

+ an additional condition at $x = \infty$.

Step 5. Analysis of optimal control problem.

Maximum principle.

Maximum principle $\Rightarrow$ E-L eigenpr. + an additional constraint on switch points.

$$-y'' = \omega^2\, y\, [\epsilon_1 + (\epsilon_2 - \epsilon_1)\chi_{\mathbb{C}_+}(y^2)],$$
$$-\mathrm{i}y'(s_\pm)/\omega = \pm\sqrt{\epsilon_\infty}\, y(s_\pm),$$

Here $\chi_{\mathbb{C}_+}(\zeta) := \begin{cases} 1 & \text{if } \operatorname{Im}\zeta > 0, \\ 0 & \text{if } \operatorname{Im}\zeta \leqslant 0. \end{cases}$

$\exists\lambda_0 \geqslant 0$ s.t. $\operatorname{Im}(\varepsilon(s)y^2(s) + \omega^{-2}(y'(s))^2) = \lambda_0$ for all $s \in [s_-, s_+]$.

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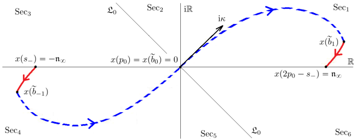
Step 6. Analysis of optimal control problem.

Extremal synthesis. Part 1.



$$[-\mathfrak{n}_2,-\mathfrak{n}_1)\overset{4}{-}\left[\overset{4}{+}(-\mathfrak{n}_2,0)\overset{3}{+}\overset{3}{-}(-\infty,-\mathfrak{n}_1)\overset{4}{-}\right]^{m_1}\overset{4}{+}\{0\}\overset{1}{+}\left[\overset{1}{-}(\mathfrak{n}_1,+\infty)\overset{6}{-}\overset{6}{+}(0,\mathfrak{n}_2)\overset{1}{+}\right]^{m_2}\overset{1}{-}(\mathfrak{n}_1,\mathfrak{n}_2];$$

$$(-\mathfrak{n}_2,-\mathfrak{n}_1]\overset{3}{+}\{\infty\}\overset{6}{+}[\mathfrak{n}_1,\mathfrak{n}_2)$$

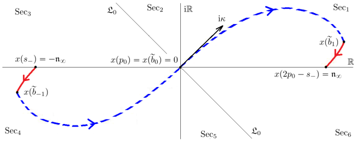
$$[-\mathfrak{n}_2,-\mathfrak{n}_1)\overset{4}{-}\{0\}\overset{1}{-}(\mathfrak{n}_1,\mathfrak{n}_2];$$

$$(-\mathfrak{n}_2,-\mathfrak{n}_1]\overset{3}{+}\left[(-\infty,-\mathfrak{n}_2)\overset{4}{-}(-\mathfrak{n}_1,0)\overset{3}{+}\right]^{m}\{\infty\}\left[\overset{6}{+}(0,\mathfrak{n}_1)\overset{1}{-}(\mathfrak{n}_2,+\infty)\right]^{m}\overset{6}{+}[\mathfrak{n}_1,\mathfrak{n}_2);$$

$$[-\mathfrak{n}_2,-\mathfrak{n}_1)\overset{4}{-}\left[(-\mathfrak{n}_1,0)\overset{3}{+}(-\infty,-\mathfrak{n}_2)\overset{4}{-}\right]^{m}\{0\}\left[\overset{1}{-}(\mathfrak{n}_2,+\infty)\overset{6}{+}(0,\mathfrak{n}_1)\right]^{m}\overset{1}{-}(\mathfrak{n}_1,\mathfrak{n}_2].$$

$$(-\mathfrak{n}_2,-\mathfrak{n}_1]\overset{3}{+}\left[\overset{3}{-}(-\infty,-\mathfrak{n}_1)\overset{4}{-}\overset{4}{+}(-\mathfrak{n}_2,0)\overset{3}{+}\right]^{m_1}\overset{3}{-}\{\infty\}\overset{6}{-}\left[\overset{6}{+}(0,\mathfrak{n}_2)\overset{1}{+}\overset{1}{-}(n_1,+\infty)\overset{6}{-}\right]^{m_2}\overset{6}{+}[\mathfrak{n}_1,\mathfrak{n}_2);$$

Extremal synthesis. Part 2.



$$(-\mathfrak{n}_2,-\mathfrak{n}_1] \overset{3}{+} \overset{3}{-} (-\infty,-\mathfrak{n}_1) \overset{4}{-} \left[\overset{4}{+} (-\mathfrak{n}_2,0) \overset{3}{+} \overset{3}{-} (-\infty,-\mathfrak{n}_1) \overset{4}{-} \right]^{m_1} \overset{4}{+} \{0\}$$

$$\overset{1}{+} \left[\overset{1}{-} (n_1,+\infty) \overset{6}{-} \overset{6}{+} (0,\mathfrak{n}_2) \overset{1}{+} \right]^{m_2} \overset{1}{-} (n_1,+\infty) \overset{6}{-} \overset{6}{+} [\mathfrak{n}_1,\mathfrak{n}_2);$$

$$[-\mathfrak{n}_2,-\mathfrak{n}_1) \overset{4}{-} \overset{4}{+} (-\mathfrak{n}_2,0) \overset{3}{+} \left[\overset{3}{-} (-\infty,-\mathfrak{n}_1) \overset{4}{-} \overset{4}{+} (-\mathfrak{n}_2,0) \overset{3}{+} \right]^{m_1} \overset{3}{-} \{\infty\}$$

$$\overset{6}{-} \left[\overset{6}{+} (0,\mathfrak{n}_2) \overset{1}{+} \overset{1}{-} (\mathfrak{n}_1,+\infty) \overset{6}{-} \right]^{m_2} \overset{6}{+} (0,\mathfrak{n}_2) \overset{1}{+} \overset{6}{-} (\mathfrak{n}_1,\mathfrak{n}_2];$$

$$[-\mathfrak{n}_2,-\mathfrak{n}_1) \overset{4}{-} \left[\overset{4}{+} (-\mathfrak{n}_2,0) \overset{3}{+} \overset{3}{-} (-\infty,-\mathfrak{n}_1) \overset{4}{-} \right]^{m_1} \overset{4}{+} \{0\} \overset{1}{+} \left[\overset{1}{-} (\mathfrak{n}_1,+\infty) \overset{6}{-} \overset{6}{+} (0,\mathfrak{n}_2) \overset{1}{+} \right]^{m_2} \overset{1}{-} (\mathfrak{n}_1,\mathfrak{n}_2];$$

Step 6. Analysis of optimal control problem.

Extremal synthesis. Part 3.

$$[-\mathfrak{n}_2,-\mathfrak{n}_1)\overset{4}{-}\overset{4}{+}(-\mathfrak{n}_2,0)\overset{3}{+}\left[\overset{3}{-}(-\infty,-\mathfrak{n}_1)\overset{4}{-}\overset{4}{+}(-\mathfrak{n}_2,0)\overset{3}{+}\right]^{m_1}_{\{\infty\}}$$

$$(-\mathfrak{n}_2,-\mathfrak{n}_1]\left[\overset{3}{+}(-\infty,-\mathfrak{n}_2)\overset{4}{-}(-\mathfrak{n}_1,0)\right]^{m_1}\overset{3}{+}\mathfrak{L}_0\overset{2}{+}\mathrm{i}\mathbb{R}_+\left[\overset{1}{-}(\mathfrak{n}_2,+\infty)\overset{6}{+}(0,\mathfrak{n}_1)\right]^{m_2}\overset{1}{-}(\mathfrak{n}_1,\mathfrak{n}_2];$$

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Step 6. Analysis of optimal control problem.

Extremal synthesis. Part 4.

$$\dots \overset{4}{-}\overset{4}{+}_{i\mathbb{R}_-}\overset{5}{+}\overset{5}{-}_{\mathfrak{L}_0}\overset{6}{-}\overset{6}{+}\dots \quad \text{and} \quad \dots\overset{4}{-}\overset{4}{+}_{i\mathbb{R}_-}\overset{5}{+}_{\mathfrak{L}_0}\overset{6}{+}\dots$$

$$(-\mathfrak{n}_2,-\mathfrak{n}_1]\left[\overset{3}{+}_{(-\infty,-\mathfrak{n}_2)}\overset{4}{-}_{(-\mathfrak{n}_1,0)}\right]^{m_1}\overset{3}{+}_{(-\infty,-\mathfrak{n}_2)}\overset{4}{-}_{i\mathbb{R}_-}\overset{5}{+}_{\mathfrak{L}_0}\left[\overset{6}{+}_{(0,\mathfrak{n}_1)}\overset{1}{-}_{(\mathfrak{n}_2,+\infty)}\right]^{m_2}\overset{6}{+}_{[\mathfrak{n}_1,\mathfrak{n}_2)};$$

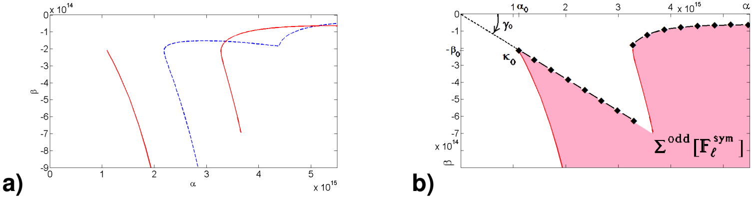
$$(-\mathfrak{n}_2,-\mathfrak{n}_1]\left[\overset{3}{+}_{(-\infty,-\mathfrak{n}_2)}\overset{4}{-}_{(-\mathfrak{n}_1,0)}\right]^{m_1}\overset{3}{+}_{(-\infty,-\mathfrak{n}_2)}\overset{4}{-}_{i\mathbb{R}_-}\overset{5}{+}_{\mathfrak{L}_0}\left[\overset{6}{+}_{(0,\mathfrak{n}_1)}\overset{1}{-}_{(\mathfrak{n}_2,+\infty)}\right]^{m_2}\overset{6}{+}_{(0,\mathfrak{n}_1)}\overset{1}{-}_{(\mathfrak{n}_1,\mathfrak{n}_2]};$$

$$[-\mathfrak{n}_2,-\mathfrak{n}_1)\left[\overset{4}{-}_{(-\mathfrak{n}_1,0)}\overset{3}{+}_{(-\infty,-\mathfrak{n}_2)}\right]^{m_1}\overset{4}{-}_{i\mathbb{R}_-}\overset{5}{+}_{\mathfrak{L}_0}\left[\overset{6}{+}_{(0,\mathfrak{n}_1)}\overset{1}{-}_{(\mathfrak{n}_2,+\infty)}\right]^{m_2}\overset{6}{+}_{[\mathfrak{n}_1,\mathfrak{n}_2)};$$

$$[-\mathfrak{n}_2,-\mathfrak{n}_1)\left[\overset{4}{-}_{(-\mathfrak{n}_1,0)}\overset{3}{+}_{(-\infty,-\mathfrak{n}_2)}\right]^{m_1}\overset{4}{-}_{i\mathbb{R}_-}\overset{5}{+}_{\mathfrak{L}_0}\left[\overset{6}{+}_{(0,\mathfrak{n}_1)}\overset{1}{-}_{(\mathfrak{n}_2,+\infty)}\right]^{m_2}\overset{6}{+}_{(0,\mathfrak{n}_1)}\overset{1}{-}_{(\mathfrak{n}_1,\mathfrak{n}_2]},$$

$$\dots\overset{3}{+}\overset{3}{-}_{\mathfrak{L}_0}\overset{2}{-}\overset{2}{+}_{i\mathbb{R}_+}\overset{1}{+}\overset{1}{-}\dots \quad \text{and} \quad \dots\overset{3}{+}_{\mathfrak{L}_0}\overset{2}{+}_{i\mathbb{R}_+}\overset{1}{+}\overset{1}{-}\dots$$

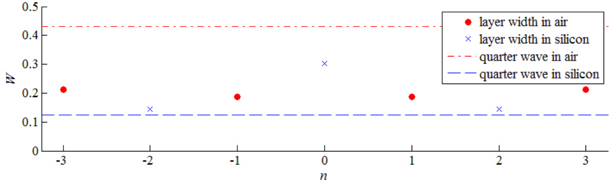
$$\dots \overset{4}{-}\overset{4}{+}_{i\mathbb{R}_-}\overset{5}{+}\overset{5}{-}_{\mathfrak{L}_0}\overset{6}{-}\overset{6}{+}\dots \quad \text{and} \quad \dots\overset{4}{-}\overset{4}{+}_{i\mathbb{R}_-}\overset{5}{+}_{\mathfrak{L}_0}\overset{6}{+}\dots$$



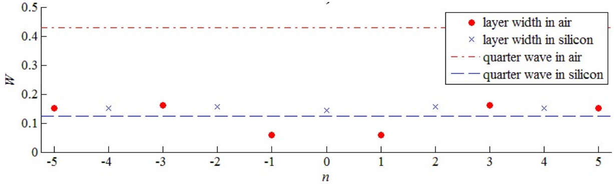
(a) The computed parts of Pareto frontiers of even-mode resonances and odd-mode resonances of minimal modulus for $\epsilon_1 = \epsilon_\infty = 1$, $\epsilon_2 = 11.97$, and $\ell = 0.1243 \cdot 10^{-6}$.

(b) The drawing of the set of achievable odd-mode resonances. The line marked '**–◆–**' shows the corresponding Pareto frontier of minimal decay. It is proved analytically that $\omega_0 = \alpha_0 - i\beta_0 = e^{\gamma_0}\rho_{\min}(\gamma_0)$ is Pareto optimal and that there exists a jump of $\rho_{\min}(\gamma)$ at this point.

Optimal symmetric structures.



a)



b)

Layers' widths for (a) the odd-mode and (b) the even-mode optimal resonators for $\omega_1 = 3.653 \cdot 10^6 - \mathrm{i} \cdot \ 3.653 \cdot 10^4$.

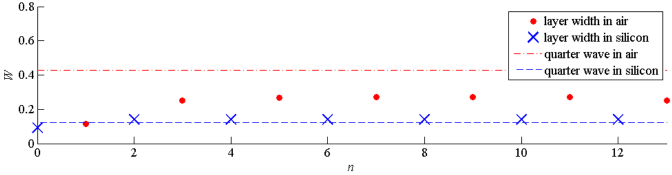
$Q = 50$, the wavelength in vacuum is $\lambda_1 = 1720$ nm (infrared range).

The optimal odd-mode resonator has $L \approx 1.389359403$ μm and $N = 7$ layers.

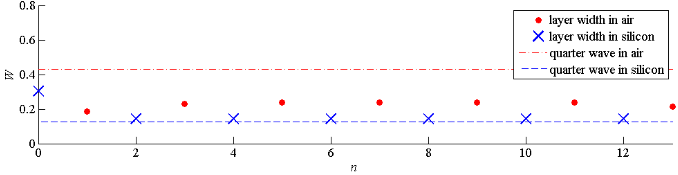
The shortest even-mode resonator has $L \approx 1.5043572352$ μm and $N = 11$.

Optimal symmetric structures, high quality-factor Q.

Much higher values of Q, but the same frequency $\mathrm{Re}\,\omega$, the right half:



a)



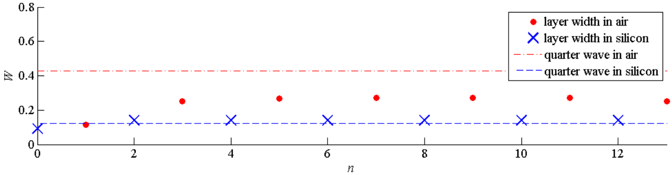
b)

Widths of layers for symmetric resonators of minimum length with:

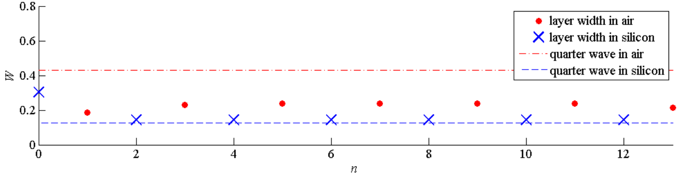
(a) $Q = 10^6$, the optimizer has an even mode;

(b) $Q = 1.1 \cdot 10^6$, the optimizer has an odd mode.

Optimal symmetric structures, high quality-factor Q.



a)



b)

Observation (K., Koch, Verbytskyi, ‘20):

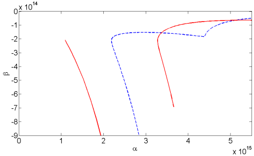
(1) gradually changing deviations from periodicity,

(2) centered defect.

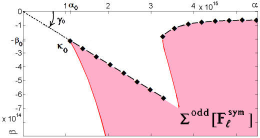
Thm (K., Koch, Verbytskyi, '20).
1) If $\epsilon_1 < \epsilon_\infty < \epsilon_2$, the reduction is complete.
Every Pareto optimal resonance/structure for Problem 1
is Pareto optimal for Problem 2.

2) If $\epsilon_1 \leqslant \epsilon_\infty \leqslant \epsilon_2$, the reduction is almost complete.
From the Pareto optimal frontier/structure for Problem 2 one can obtain
the Pareto optimal frontier for Problem 1 and at least some of optimal structures.

Proof. Continuity/lower semi-continuity of $\rho_{\min}(\cdot)$ and star-like resonance-free region.



a)



b)

(a) Pareto frontiers of even-mode resonances and odd-mode resonances of minimal modulus for $\epsilon_1 = \epsilon_\infty = 1$, $\epsilon_2 = 11.97$, and $\ell = 0.1243 \cdot 10^{-6}$.

(b) The set of achievable odd-mode resonances. The line marked '**–⬩–**' shows the corresponding Pareto frontier of minimal decay. It is proved analytically that $\omega_0 = \alpha_0 - \mathrm{i}\beta_0 = e^{\gamma_0}\rho_{\min}(\gamma_0)$ is Pareto optimal and that $\exists$ a jump of $\rho_{\min}(\gamma)$ at this point.

I.K. is grateful to

- the EU-financed projects AMMODIT (grant agreement MSCA-RISE-2014-645672-AMMODIT),
- the Alexander von Humboldt Foundation,
- the VolkswagenStiftung,
- and German Research Foundation (DFG, Heisenberg-Program, Projektnummer: 509859995)

for the financial support during various stages of this reasearch.

Thank you very much for your attention!

Many thanks to the organizers for the hospitality of Reims

and a very special conference!