Inverse resonance scattering on rotationally symmetric manifolds

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Abstract. We discuss inverse resonance scattering for the Laplacian on a rotationally symmetric manifold $M = (0, \infty) \times Y$ whose rotation radius is constant outside some compact interval. The Laplacian on M is unitarily equivalent to a direct sum of one-dimensional Schrödinger operators with compactly supported potentials on the half-line. We prove

• Asymptotics of counting function of resonances at large radius.

• The rotation radius is uniquely determined by its eigenvalues and resonances.

• There exists an algorithm to recover the rotation radius from its eigenvalues and resonances.

The proof is based on some non-linear real analytic isomorphism between two Hilbert spaces.

Historical review There is an abundance of works devoted to the spectral theory and inverse problems for the surface of revolution from the view points of classical inverse Strum-Liouville theory, integrable systems, micro-local analysis, see Aberra-Agrawal [98] ,.... For integrable systems associated with surfaces of revolution, see e.g. Konopelchenko-Taimanov [96], Sanders-Wang [03],.....

Isozaki-Korotyaev [17] solved the inverse spectral problem for rotationally symmetric manifolds (finite perturbed cylinders), which includes a class of surfaces of revolution, by giving an analytic isomorphism from the space of spectral data onto the space of functions describing the radius of rotation. In another paper Isozaki-Korotyaev [19] studied inverse problems for Laplacian on the torus. Moreover, they obtained stability estimates: the spectral data in terms of the profile (the radius of the rotation) and conversely, the profile in term of the spectral data.

Resonances for specific cases of surfaces of revolution are discussed by Christiansen [04], Datchev- Kang-Kessler [15], Datchev-Hezari [13], Datchev [16]. As far as the authors know, the results in our talk about resonances for Laplacian on surfaces of revolution for our case are new.

Resonances for the case of $\mathbb{R}^d$. Consider the Schrödinger operator $H = -\Delta + V$ on $L^2(\mathbb{R}^d)$, where $d \geqslant 1$ is odd and the potential V is real compactly supported. Define the resolvent $R(\lambda) = (H - \lambda)^{-1}$, which meromorphic in the cut domain $\Lambda_1 = \mathbb{C} \setminus [0, \infty)$ with a finite number of poles on $(-\infty, 0]$. Each pole is eigenvalue. We introduce the two-sheeted Riemann surface Λ of $\sqrt{\lambda}$ obtained by joining the upper and lower rims of two copies of the cut plane $\mathbb{C} \setminus [0, \infty)$ in the usual (crosswise) way. Consider the function $F(\lambda) = (R(\lambda)f, g)$ on the first sheet Λ_1, where $f, g \in L^2(\mathbb{R}^d)$ are compactly supported. The function F has a meromorphic extension from the first sheet Λ_1 on the second sheet Λ_2 of the Riemann surface $\sqrt{\lambda}$. Each pole on the second sheet Λ_2 is a resonance for H.

Resonances for the multidimensional case ($d \geqslant 2$) were studied by Melrose, Sjöstrand, and Zworski and other. We describe the main results for odd d:

$$Cr^{\frac{d}{2}} \leqslant \mathcal{N}(r) \leqslant C_* r^{\frac{d}{2}} \qquad \forall \ r \geqslant 1,$$

for some constants C, C_* depending from V, d only and $\mathcal{N}(r)$ is the number of resonances in $\mathbb{C}$ having modulus $\leqslant r$ and counted according to multiplicity.

Even case is more complicated since the Riemann surface $\sim \log \lambda$.

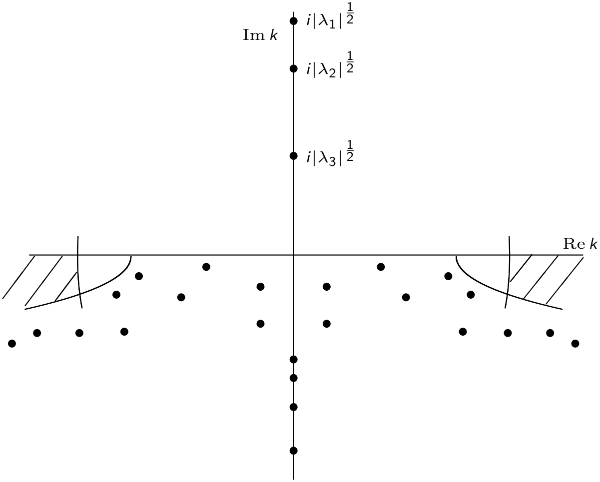

Figure: Resonances and eigenvalues $\lambda_1 < \lambda_2 < \ldots$ for the Schrödinger operator H. The resonances are marked by circles. The forbidden domain for the resonances is shaded, where there are No resonances.

A lot of papers are devoted to resonances of one-dimensional Schrödinger operators, for example, Froese [97], Hitrik [99], Korotyaev [04], Simon [00], Zworski [87] and so on. Inverse problems (uniqueness, reconstruction, characterization) in terms of resonances were solved by Korotyaev for a Schrödinger operator with a compactly supported potential on the real line [05] and the half-line [04]. Note that Zworski [02], Brown-Knowles-Weikard [03] discussed the uniqueness for the inverse resonance problem.

There are results about particles in external electric fields. For example. The resonances for one-dimensional operators $-\frac{d^2}{dx^2} + V_\pi + V$, where V_π is periodic and V is a compactly supported potential were considered by Firsova [84], Korotyaev [11]. Christiansen [06] considered resonances and the inverse resonance problem for steplike potentials. Resonances for Stark operator perturbed by compactly supported potentials were discussed by Korotyaev [17] and by Froese and Herbst [19]. For Stark operators resonances determine the potential uniquely (due to EK [17]), similar to the case of Schrödinger operator with a compactly supported potential on the half-line.

Resonances for 1-dim Dirac operators with a compactly supported potential are discussed by and Korotyaev with co-authors Ianchenko and Mokeev. Inverse problems terms of resonances (uniqueness, reconstruction, characterization) for Dirac operators with a compactly supported potential on the half line was solved by EK+Mokeev on the real line [23] and the half-line [20].

Resonances for three and fourth order differential operators with compactly supported coefficients on the line were studied by Badanin and Korotyaev [19]. Here inverse resonance problems are still open. In fact here there are a lot of open problems.

Finally I discuss so-called stability problems for resonances. For example, let k_o be a resonance for $H = -\Delta + V$ in $\mathbb{R}^d, d \geqslant 1$, where V is a compactly supported potential. There is a question: the point $k_\varepsilon = k_o + \varepsilon$ (where $\varepsilon \in \mathbb{C}$ is very small) is a resonance for some compactly supported potential? Such problem was solved only for Schrödinger operator (EK [04]) and Dirac operators (Mokeev [22]) with a compactly supported potential on the half-line. But for $d \geqslant 2$ this problem is still open.

Note that there are other stability problems. Some of them were considered by Marletta, Shterenberg, Weikard [10].

Recall results from EK [04], which are used below. Consider the operator $Ty = -y'' + py$, $y(0) = 0$ on $\mathbb{R}_+$. Assume that $p \in \mathcal{P}$, where

$$\mathcal{P} = \left\{ g \in L^1_{real}(\mathbb{R}_+), \quad \mathrm{supp}\, g \subset [0,1], \quad \sup \mathrm{supp}\, g = 1 \right\}.$$

It is known that T has purely abs. cont. spectrum $[0, \infty)$ plus a finite number $m \geqslant 0$ of eigenvalues $\lambda_1 < ... < \lambda_m < 0$ below the continuum. The Jost solution $f_+(x,k)$ is a solution to the equation $-f_+'' + pf_+ = k^2 f_+, x \geqslant 0$ at $k \in \mathbb{C} \setminus \{0\}$ under the condition $f_+(x,k) = e^{ixk}$, $x \geqslant 1$. The Jost function $\psi(k)$ is defined by $\psi(k) = f_+(0,k)$. Let $n_+(f)$ be the number of zeros of f analytic in $\mathbb{C}_+$, each zero being counted according to its multiplicity. For a function f analytic in a neighborhood of 0, we define $n_o(f) = \mathfrak{s}$, if $f(z) = z^{\mathfrak{s}} g(z)$ where $g(0) \neq 0$. The Jost function $\psi(k)$ is entire on $\mathbb{C}$ and has the following asymptotics uniformly in $\arg k \in [0, \pi]$:

$$\psi(k) = 1 + O(1/k) \qquad \text{as} \quad |k| \to \infty. \tag{1}$$

We recall known facts about entire functions. An entire function f is said to be of *exponential type* if there is a constant α such that $|f(z)| \leqslant \text{const.}\ e^{\alpha|z|}$ everywhere. The function f is said to belong to the Cartwright class E_{Cart}, if f is entire, of exponential type, and satisfies:

$$\int_{\mathbb{R}} \frac{\log(1+|f(x)|)dx}{1+x^2} < \infty, \quad \rho_+(f) = 0, \quad \rho_-(f) = 2,$$

where $\rho_\pm(f) = \limsup_{y\to\infty} \frac{\log|f(\pm iy)|}{y}$.

We recall the well known result of Levinson.

Theorem (Levinson). *Let the entire function $f \in E_{Cart}$. Then*

$$\mathcal{N}(r, f) = \frac{2}{\pi} r + o(r), \quad \textit{as} \quad r \to \infty,$$

where $\mathcal{N}(r, f)$ is the total number of zeros of f with modulus $\leqslant r$.

Let $f \in E_{Cart}$ and denote by $\{z_n\}_{n=1}^{\infty}$ the sequence of its zeros $\neq 0$ (counted with multiplicity), so arranged that $0 < |z_1| \leqslant |z_2| \leqslant \dots$. Then the Hadamard factorization holds

$$f(z) = z^{\mathfrak{s}} C e^{iz} \lim_{r \to +\infty} \prod_{|z_n| \leqslant r} \left(1 - \frac{z}{z_n}\right), \qquad C = \frac{f^{(\mathfrak{s})}(0)}{\mathfrak{s}!},$$

for some integer $\mathfrak{s}$, where the product converges uniformly in every bounded disc and

$$\sum_{1}^{\infty} \frac{|\operatorname{Im} z_n|}{|z_n|^2} < \infty,$$

$$\frac{f'(z)}{f(z)} = i + \frac{\mathfrak{s}}{z} + \lim_{r \to \infty} \sum_{|z_n| \leqslant r} \frac{1}{z - z_n}$$

uniformly on compact subsets of $\mathbb{C} \setminus \left(\{0\} \cup \bigcup \{z_n\}\right)$.

- The S-matrix is defined by

$$S(k) = \frac{\psi(-k)}{\psi(k)} = e^{-i2\phi_{sc}(k)}, \quad \phi_{sc}(k) = \arg \psi(k), \quad k \in \mathbb{R}. \tag{2}$$

We call ϕ_{sc} phase shift. Note that $S(k)$ is meromorphic on $\mathbb{C}$. We can define a resonance of T as a pole of $S(k)$ in $\mathbb{C}_-$.

- The function ψ in $\mathbb{C}_+$ has simple zeros given by

$$k_1 = i|\lambda_1|^{\frac{1}{2}}, \dots k_m = i|\lambda_m|^{\frac{1}{2}} \in i\mathbb{R}_+, \tag{3}$$

possibly one simple zero at 0 and infinite numbers in $\mathbb{C}_-$. For the zeros in $\mathbb{C}_- \cup \{0\}$ we have $0 \leqslant |k_{m+1}| \leqslant |k_{m+2}| \leqslant \dots$ (counted with multiplicity). The zeros of ψ in $\mathbb{C}_-$ are called the resonances.

Introduce the set $\mathcal{J}$ of all possible Jost functions from EK [04].
Definition $\mathcal{J}$. *By $\mathcal{J}$ we mean the class of all entire functions f having the form*

$$f(k) = 1 + \frac{1}{2ik}\Big(\hat{F}(k) - \hat{F}(0)\Big), \qquad k \in \mathbb{C}, \tag{4}$$

where $\hat{F}(k) = \int_0^1 F(x)e^{2ixk}dx$ and $F \in \mathcal{P}$. In addition the sequence $(k_n)_{n=1}^{\infty}$ of zeros of f satisfies:
i) all zeros of the function f in $\overline{\mathbb{C}}_+$ are simple, belong to $i\overline{\mathbb{R}}_+$.
ii) all zeros $k_1, .., k_m, m \geqslant 0$ of the function f in $\mathbb{C}_+$ satisfy

$$\begin{aligned} &|k_1| > |k_2| > ... > |k_m| > 0, \\ &(-1)^j f(-k_j) > 0, \qquad j = 1, ..., m. \end{aligned} \tag{5}$$

Note that $\mathcal{J} \subset E_{Cart}$. We sometimes write $f_+(x,k,p), \psi(k,q), ...$ instead of $f_+(x,k), \psi(k), ...$, when several potentials are being dealt with.

We define the mapping $\psi : \mathcal{P} \to \mathcal{J}$ by $p \to \psi(\cdot, p)$, i.e., each $p \in \mathcal{P}$ maps the Jost function $\psi(\cdot, p) \in \mathcal{J}$. We recall the main result on the inverse resonance scattering from EK [04].

Theorem A. *i) The mapping $p \to \psi(\cdot, p)$ from $\mathcal{P}$ to $\mathcal{J}$ is a bijection.*

ii) If the Jost function $\psi(\cdot, p)$ is given for some $p \in \mathcal{P}$. Then there exists the algorithm to recover the potential $p \in \mathcal{P}$.

Note that $\psi \in \mathcal{J} \subset E_{Cart}$. There are a lot of results about such functions. For example, the Levinson Theorem gives

$$\mathcal{N}(r, \psi) = \frac{2}{\pi} r + o(r) \quad \textit{as} \quad r \to \infty,$$

where $\mathcal{N}(r, \psi)$ is a number of zeros of ψ with modulus $\leqslant r$. Recall that Zworski [87] proved it in 1987 directly (without the Levinson Theorem). The Fredholm determinant for Schrödinger operators and Dirac operators with compactly supported potentials $\in E_{Cart}$. For Stark operators and for three and fourth order differential operators the corresponding Fredholm determinants do no belong to E_{Cart}. It creates a big problem to solve inverse resonance problem.

Rotationally symmetric manifolds. We consider a rotationally symmetric manifold $M = \mathbb{R}_+ \times Y$ equipped with a warped product metric

$$g = (dx)^2 + r^2(x) g_Y, \quad x \geqslant 0. \tag{6}$$

Here (Y, g_Y) is a compact m-dimensional Riemannian manifold (with or without boundary), called transversal manifold, and $r(x) > 0$ is the rotation radius given by

$$r = r_o\, e^{\frac{2}{m} Q}, \qquad Q(x) = -\int_x^1 q(t)dt, \quad x \in [0,1],$$
$$r(x) = r_o = \mathrm{const} > 0 \quad \text{for} \quad x \geqslant 1.$$

Assume that q belongs to a class $\mathcal{P}_1$ given by

$$\mathcal{P}_1 = \left\{ g, g' \in L^1_{real}(\mathbb{R}_+), \quad \operatorname{supp} g \subset [0,1], \quad \sup \operatorname{supp} g = 1 \right\}.$$

Below we show that the geometry (i.e., the rotation radius r and hence all derived quantities up to two integration constants) is determined by q.

The Laplacian Δ_M on M has the form

$$-\Delta_M = -\frac{1}{r^m}\partial_x(r^m\partial_x) - \frac{\Delta_Y}{r^2},$$

where Δ_Y is the Laplacian on Y. We assume the Dirichlet boundary condition, i.e. the domain of Δ_M consists of the functions $f = f(x,y), (x,y) \in M = \mathbb{R}_+ \times Y$, satisfying the following boundary condition

$$f|_{\partial M} = 0. \tag{7}$$

The negative Laplacian $-\Delta_Y$ on Y (with suitable boundary conditions when Y has a boundary) has a discrete spectrum, $0 \leqslant E_1 \leqslant E_2 \leqslant \cdots$, with corresponding orthonormal family of eigenfunctions $\Psi_\nu, \nu \geqslant 1$, in $L^2(Y)$. Recall simple examples.
(1) The case when Y has no boundary. For example, if $Y = \mathbb{S}^1$, then $E_1 = 0, E_2 = \pi^2, \cdots$.
(2) The case when Y has a boundary. For example, if Y is a compact interval in $\mathbb{R}^1$, e.g. $[0,1]$, then $E_1 = 0, E_2 = \pi^2, \cdots$, for the Neumann boundary condition, and $E_1 = \pi^2, E_2 = (2\pi)^2, \cdots$, for the Dirichlet boundary condition.

By the spectral decomposition of $-\Delta_Y$, the Laplacian on (M, g) acting on $L^2(M)$ is unitarily equivalent to a direct sum of one-dimensional Schrödinger operators H_ν, namely,

$$-\Delta_M \backsimeq \oplus_{\nu=1}^{\infty} (H_\nu + u_{\nu,0}). \tag{8}$$

Here the operator H_ν on $L^2(\mathbb{R}_+)$ is given by

$$H_\nu f = -f'' + p_\nu f, \quad f(0) = 0, \tag{9}$$

and the potential $p = p_\nu(x)$ is defined by

$$\begin{cases} p = P(q) = q' + q^2 + u_\nu - u_{\nu,0}, \\ u_\nu(x) = u_{\nu,0} e^{-\frac{4}{m} Q(x)}, \quad Q(x) = -\int_x^1 q(t)dt, \\ u_{\nu,0} = \frac{E_\nu}{r^2(1)} = \mathrm{const}\,. \end{cases} \tag{10}$$

Note that

$$\operatorname{supp} p_\nu \subset [0, 1], \quad p_\nu \in L^1(0, 1).$$

Below we fix an index ν, and omit it. We consider the operator $Hf = -f'' + pf, f(0) = 0$, where the potential $p = P(q) = q' + q^2 + u_\nu - u_{\nu,0}$ is given by (10). The operator H has purely abs. cont. spectrum $[0, \infty)$ plus a finite number $m = m_\nu \geqslant 0$ of negative eigenvalues $\lambda_1 < ... < \lambda_m < 0$.

Theorem 1. *i) If p is given by (10) for some $q \in \mathcal{P}_1$, then $p \in \mathcal{P}$.*
ii) If $r(x) \leqslant r_o = r(1)$ for all $x \in [0, 1]$, then the Laplacian $-\Delta_M$ has no eigenvalues.
iii) There exists a radius $r(x) > r_o$ for all $x \in (0, 1)$ and ν_o large enough such that each operator $H_\nu, \nu > \nu_o$ has eigenvalues and the Laplacian Δ_M has infinite number of eigenvalues.

Remark. It is important for resonance scattering to show that $p \in \mathcal{P}$ for each $q \in \mathcal{P}_1$. In order to show i) we need the following support property of the non-linear mappings $q \to p_\nu$, which plays a key role.

Lemma C. *Let $q, q' \in L^1(\Omega)$ for $\Omega = (1-\tau, 1)$ for some $\tau > 0$ and $q(1) = 0$. Assume that q satisfies on Ω the following equation*

$$q' + q^2 + u - u_0 = 0, \tag{11}$$

where $u = u_0 e^{\beta Q}$ with $\beta, u_0 > 0$. Then $q = 0$ on $\omega = (1-\varepsilon^2, 1)$ for $\varepsilon > 0$ small enough.

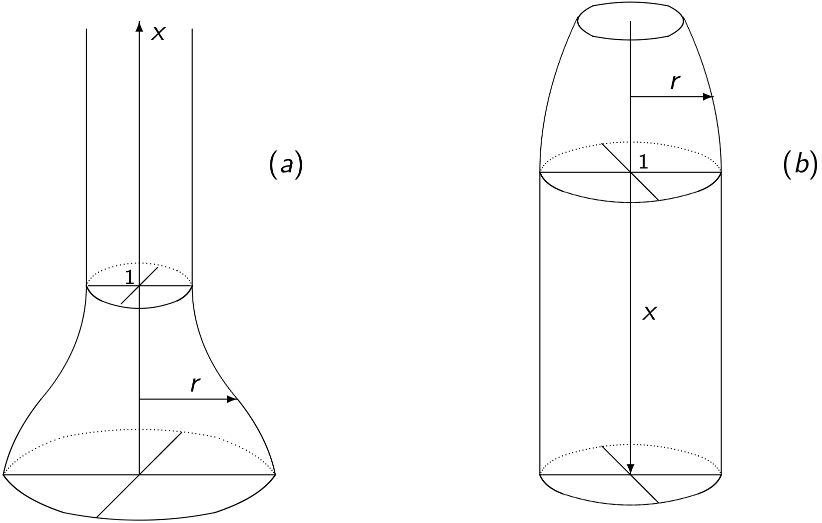

Figure: (a) Δ has eigenvalues, (b) Δ has not eigenvalues

From $P(q) \in \mathcal{P}$ and results of Zworski [87] we obtain
Corollary 2. *Let $q \in \mathcal{P}_1$. Then*

$$\mathcal{N}_r(\psi) = \frac{2r}{\pi}(1 + o(1)) \qquad \text{as} \quad r \to \infty. \tag{12}$$

Remark. The asymptotic behavior of the distribution function of the eigenvalues of the Laplacian on warped product manifolds with cylindrical ends are discussed in Christiansen-Zworski [95].

Inverse Resonance Problem. We discuss now inverse resonance problems. We show that all resonances determine the surface (or the rotation radius) uniquely. It is a first result about inverse resonance problems for Laplacian and the Riemann surface.

We sometimes write $\psi(k,q), k_n(q), \cdots$ instead of $\psi(k), k_n, \cdots$, when several functions are dealt with. We introduce the Sobolev space of real functions

$$W = \Big\{ q, q' \in L^2_{real}(\mathbb{R}_+),\ \operatorname{supp} q \subset (0,1),\ q(0) = q(1) = 0 \Big\},$$

$$\|q\|_W^2 = \|q'\|^2 = \int_0^1 |q'(x)|^2 dx.$$

The main result of my talk is

Theorem 3. *i) Let ψ_j be the Jost function for $q_j \in W \cap \mathcal{P}_1, j = 1,2$. If $\psi_1 = \psi_2$, then $q_1 = q_2$.*
ii) Let S_j be the S-matrix for $q_j \in W \cap \mathcal{P}_1, j = 1,2$. If $S_1 = S_2$, then $q_1 = q_2$.
iii) Any $q \in W \cap \mathcal{P}_1$ is uniquely determined by its eigenvalues and resonances. Moreover, there exists an algorithm to recover q from its eigenvalues and resonances.

We discuss the Proof of Theorem 3. We have two Jost solutions $f_+(x,k,p_j), p_j = P(q_j), j = 1,2$ for some $q_j \in \mathcal{P}_1$ to the equation

$$-f_+'' + pf_+ = k^2 f_+, \quad x \geqslant 0, \quad k \in \mathbb{C} \setminus \{0\} \qquad (13)$$

satisfying the condition $f_+(x,k,p_j) = e^{ixk}, \ x \geqslant 1$. These Jost functions satisfy

$$f_+(0,k,p_1) = f_+(0,k,p_2),$$

where $p = P(q)$ is defined in terms of the function q by

$$\begin{cases} p = P(q) = q' + q^2 + u_\nu - u_{\nu,0}, \\ u_\nu(x) = u_{\nu,0} e^{-\frac{4}{m}Q(x)}, \quad Q(x) = -\int_x^1 q(t)dt, \\ u_{\nu,0} = \frac{E_\nu}{r^2(1)}. \end{cases}$$

Let $k_n(p), n \geqslant 1$ be the zeros of the Jost function $f_+(0,k,p)$ for the equation $-y'' + py$. Recall that due to Theorem A the mapping

$$p \mapsto (k_n(p))_{n=1}^{\infty}, \quad p \in \mathcal{P}$$

is a bijection between $\mathcal{P}$ and some class of the zeros of the Jost functions.

We consider the case $p_j = P(q_j)$, and recall that $k_n(P(q_j)), n \geqslant 1$ be the zeros of the Jost function $f_+(0,k,P(q))$ for the equation $-y'' + P(q)y$. Then we obtain the identity

$$(k_n(P(q_1))_1^{\infty} = (k_n(P(q_2))_{n=1}^{\infty}, \qquad q_1, q_2 \in W_1^0.$$

The mapping $q \to (k_n(P(q)))_1^{\infty}$ is the composition of two mappings $q \to P(q)$ and $p \to (k_n(p))_1^{\infty}$, where each of them is the corresponding injection (see Theorems A, B and Lemma C). Then the mapping

$$q \mapsto (k_n(P(q))_{n=1}^{\infty},$$

is injection. We need to study the mapping $q \to P(q)$.

We introduce the Sobolev spaces $\mathscr{H}$ of real functions

$$\mathscr{H} = \left\{ q \in L^2_{real}(0,1) : \int_0^1 q(x)dx = 0 \right\}$$

equipped with norm $\|q\|^2_{\mathscr{H}} = \int_0^1 |q(x)|^2 dx$. Thus we can consider the mapping $V : W \to \mathscr{H}$ given by

$$\begin{aligned} v &= V(q) = q' + q^2 + u - c_0, \quad u(Q) = u_0 e^{-\beta Q}, \\ Q(x) &= -\int_x^1 q(t)dt, \quad c_0 = \int_0^1 (q^2 + u)dx, \end{aligned} \tag{14}$$

where $u_0 = \frac{E_\nu}{r_o^2}$ and $\beta = \frac{4}{m}$. We need to the following result on the mapping $q \to v = V(q)$.

Theorem B *The mapping $V : W \to \mathscr{H}$, given by (14) is a real analytic isomorphism between the Hilbert spaces W and $\mathscr{H}$ and satisfies:*

$$\|q'\|^2 \leqslant \|v\|^2 \leqslant \|q'\|^2 + 2\|q\|^3\|q'\| + C_*\|q\|^2 e^{2\beta\|q\|}, \tag{15}$$

where the constant $C_* = u_0(\beta + 1)(2 + \beta u_0)$.

In order to prove this theorem we use results about an isomorphism between two Hilbert spaces. There are various methods to prove an isomorphism between two Hilbert spaces. We use on of them from the paper of Kargaev and Korotyaev [97]. We shortly describe this approach based on nonlinear functional analysis. Suppose that H, H_1 are real separable Hilbert spaces. The derivative of a map $f : H \to H_1$ at a point $y \in H$ is a bounded linear map from H into H_1, which we denote by $f'(y)$. A map $f : H \to H_1$ is compact on H, if it maps a weakly convergent sequence in H into a strongly convergent sequence in H_1. A map $f : H \to H_1$ is a real analytic isomorphism between H and H_1, if f is bijective and both f and f^{-1} are real analytic maps. Let H_C be the complexification of the real Hilbert space H.

We formulate the result from [97].

Theorem D. *Let H, H_1 be real separable Hilbert spaces equipped with norms $\|\cdot\|, \|\cdot\|_1$. Suppose that a map $f : H \to H_1$ satisfies the following conditions:*
i) f is real analytic,
ii) the operator $f'(q)$ has an inverse for all $q \in H$,
iii) there is a nondecreasing function $F : [0,\infty) \to [0,\infty)$, such that $F(0) = 0$ and

$$\|q\| \leqslant F(\|f(q)\|_1) \qquad \forall\ q \in H,$$

iv) there exists a basis $\{e_n\}_{n\in\mathbb{Z}}$ of H_1 such that each map $(f(\cdot), e_n)_1 : H \to \mathbb{R}, n \in \mathbb{Z}$, is compact,
v) for each $\varepsilon > 0$ the set $\{q : \sum n^2 (f(q), e_n)_1^2 < \varepsilon\}$ is compact.
Then f is a real analytic isomorphism between H and H_1.

Thus we study the properties of our mapping ψ and check all conditions from Theorem D.

Finally in order to show the injection of the mapping ψ we need also Lemma C, about the support of the function $P(q)$. These results help us to prove uniqueness. But it is enough to show that mapping $p = P(q), q \in \mathcal{P}_1$ maps $\mathcal{P}_1$ onto $\mathcal{P}$. It is still open problem.

Finally, we remark that there are at least 2 interesting open problems connected with our results.

We list interesting open problems associated with the theory of resonances:

1) To prove the Levinson Theorem for entire functions of any order. Though perhaps least physical, this seems to be the deepest problem mathematically.

2) To determine the second term in the asymptotics of the Levinson Theorem.