ON THE ORIGIN OF MINNAERT RESONANCES

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Introduction. Subwavelength resonant systems

- *The Minnaert bubble.*

A simple heuristic model for the dynamics of a gas bubble

M. Minnaert: On musical air-bubbles and the sounds of running water, Philosophical Magazine Series 7, 1933

$$\text{bubble radius} \ll \text{acoustic wavelenght}$$

$$\Downarrow$$

there is a (generalized) eigenfunction corresponding to a specific value ω_M of the frequency (nowadays called "Minnaert resonance") having a peak in its profile

$$\Downarrow$$

enhancement of the scattering of sound waves with frequency $\omega = \omega_M$

• *Subwavelength resonance systems.*

Models presenting Minnaert-like resonances created by the presence small inhomogeneities in an otherwise homogeneous medium have received a lot of attention in recent years, creating the new subject of

"subwavelength resonance systems"

Wave propagation in the presence of small scaled but highly contrasted inhomogeneities appears in different areas of applied sciences

acoustics (micro-bubbles)

electromagnetism (nano-particles)

elasticity (micro-inclusions)

It is observed, both experimentally and theoretically, that there is a critical ratio between the size and the contrast of the inhomogeneities under which the generated fields can be drastically enhanced, the amplification being more pronounced when the incident frequency is close to a specific value.
This enhancement has applications for the design of wave systems in a large variety of applications such as

superresolution

sensing

focusing

cloaking

the design of negative refractive index metamaterials

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The mathematical model.

Let $\Omega \subset \mathbb{R}^3$ be an open bounded and connected domain with a regular boundary $\Gamma := \partial\Omega$.
We define the contracted domain modeling the small inhomogeneity placed at the point y_0 by

$$\Omega^\varepsilon := \{x : x = y_0 + \varepsilon\ (y - y_0)\ ,\ y \in \Omega\}\ , \qquad \varepsilon \ll 1$$

and denote with $\Gamma^\varepsilon := \partial\Omega^\varepsilon$ its boundary.
The acoustic medium is defined by the density ρ and the bulk modulus k and the evolution equation for the acoustic field is

$$\frac{1}{k}\,\partial_{tt}u = \nabla\cdot\left(\frac{1}{\rho}\nabla u\right) \qquad \left[\quad \equiv \quad \partial_{tt}u = v^2\rho\nabla\cdot\left(\frac{1}{\rho}\nabla u\right) \quad\right].$$

In order to model the presence of the inhomogeneity, ρ and k are both chosen discontinuous across the boundary Γ^ε.

We consider the case of an inhomogeneity with high contrast of both is mass density and bulk modulus:

$$\frac{1}{\rho} = \frac{1}{k} = \begin{cases} \varepsilon^{-2} & \text{inside } \Omega^\varepsilon \\ 1 & \text{outside } \Omega^\varepsilon \end{cases} \equiv 1_{\mathbb{R}^3 \setminus \Omega^\varepsilon} + \varepsilon^{-2} 1_{\Omega^\varepsilon}$$

Then, one looks for the scattering solutions of the kind

$$u(t, x) = e^{-i\omega t} u_\omega(x),$$

generated by the incoming plane waves

$$u^{inc}(t, x) = e^{-i\omega t} u_\omega^{inc}(x), \qquad u_\omega^{inc}(x) = e^{i\omega\theta\cdot x}.$$

Imposing the natural transmission-type boundary conditions at Γ^ε and the outgoing Sommerfeld radiation condition at infinity, one obtains the stationary scattering boundary value problem (s.b.v.p. for short)

$$\begin{cases} \nabla\cdot(1_{\mathbb{R}^3\backslash\Omega^\varepsilon} + \varepsilon^{-2}1_{\Omega^\varepsilon})\nabla u_\omega + \omega^2(1_{\mathbb{R}^3\backslash\Omega^\varepsilon} + \varepsilon^{-2}1_{\Omega^\varepsilon})u_\omega = 0, \text{ in } \mathbb{R}^3\backslash\Gamma^\varepsilon, \\ \gamma_0^{\mathrm{ex}}(\varepsilon)u_\omega = \gamma_0^{\mathrm{in}}(\varepsilon)u_\omega\,, \qquad \gamma_1^{\mathrm{ex}}(\varepsilon)u_\omega = \varepsilon^{-2}\gamma_1^{\mathrm{in}}(\varepsilon)u_\omega\,, \qquad \text{on } \Gamma^\varepsilon, \\ \lim_{|x|\nearrow\infty}\,(x\cdot\nabla - i\omega|x|)\,u_\omega^{scatt}(x) = 0, \quad u_\omega^{scatt} := u_\omega - u_\omega^{inc}\,. \end{cases}$$

Here $\gamma_0^{\mathrm{in/ex}}(\varepsilon)$ and $\gamma_1^{\mathrm{in/ex}}(\varepsilon)$ denote the lateral Dirichlet and Neumann traces on the boundary Γ^ε.

If $[\gamma_0(\varepsilon)]$ and $[\gamma_1(\varepsilon)]$ denotes the jumps of the traces across the boundary, the above scattering problem rephrases, omitting for brevity the Sommerfeld radiation condition, as

$$\begin{cases} \left(\Delta + \omega^2\right) u_\omega = 0\,, \quad \text{in } \mathbb{R}^3 \backslash \Gamma^\varepsilon\,, \\ \\ [\gamma_0(\varepsilon)]\, u_\omega = 0\,, \qquad [\gamma_1(\varepsilon)]\, u_\omega = \left(\varepsilon^{-2} - 1\right) \gamma_1^{\text{in}}(\varepsilon) u_\omega\,, \quad \text{on } \Gamma^\varepsilon\,. \end{cases}$$

We introduce the Dirichlet-to-Neumann operator

$$DN_z(\varepsilon) : H^{1/2}(\Gamma^\varepsilon) \to H^{-1/2}(\Gamma^\varepsilon)$$

for the domain Ω^ε defined by

$$DN_z(\varepsilon)\varphi := \gamma_1^{\text{in}}(\varepsilon)\, u_\varphi\,, \quad \begin{cases} \left(\Delta + z^2\right) u_\varphi = 0\,, & \text{in } \Omega^\varepsilon\,, \\ \gamma_0^{\text{in}}(\varepsilon) u_\varphi = \varphi & \text{on } \Gamma^\varepsilon\,. \end{cases}$$

Such a definition is well-posed whenever $z^2 \notin \sigma(-\Delta^D_{\Omega^\varepsilon})$, where $\Delta^D_{\Omega^\varepsilon}$ is the Dirichlet Laplacian in $L^2(\Omega^\varepsilon)$.
Since $\inf \sigma(-\Delta^D_\Omega) > 0$, by

$$z^2 \in \sigma(-\Delta^D_{\Omega^\varepsilon}) \quad \Longleftrightarrow \quad \varepsilon^2 z^2 \in \sigma(-\Delta^D_\Omega)\,,$$

there follows that for each $z \in \mathbb{C}$ there exists $\varepsilon_0 > 0$ small enough (depending on z) such that $DN_z(\varepsilon)$ exists for all $0 < \varepsilon < \varepsilon_0$.

Assuming $\varepsilon^2\omega^2 \notin \sigma(-\Delta_\Omega^D)$, for the solution u_ω of the s.b.v.p. one has

$$\gamma_1^{\rm in}(\varepsilon)u_\omega = DN_\omega\,(\varepsilon)\,\gamma_0^{\rm in}(\varepsilon)u_\omega \equiv DN_\omega(\varepsilon)\gamma_0(\varepsilon)u_\omega.$$

and the s.b.v.p. recasts as

$$\begin{cases} \left(\Delta+\omega^2\right)u_\omega = 0\,, \qquad \text{in } \mathbb{R}^3\backslash\Gamma^\varepsilon\,, \\ \\ [\gamma_0(\varepsilon)]\,u_\omega = 0\,, \qquad [\gamma_1(\varepsilon)]\,u_\omega = \left(\varepsilon^{-2}-1\right)DN_\omega(\varepsilon)\gamma_0(\varepsilon)u_\omega, \text{ on } \Gamma^\varepsilon\,. \end{cases}$$

Hence, the initial scattering problem reduces to the search of the generalized eigenfunctions of the linear operator

$$H_\omega(\varepsilon) : \mathrm{dom}(H_\omega(\varepsilon)) \subset L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$$

defined by

$$H_\omega(\varepsilon)u := \Delta_{\mathbb{R}^3 \backslash \Gamma^\varepsilon} u,$$

$$\begin{aligned} \mathrm{dom}(H_\omega(\varepsilon)) := \Big\{ & u \in H^2_\Delta(\mathbb{R}^3 \backslash \Gamma) \cap H^1(\mathbb{R}^3) : \\ & [\gamma_1(\varepsilon)]\, u = \left(\varepsilon^{-2} - 1\right) DN_\omega(\varepsilon)\gamma_0(\varepsilon)u \Big\}, \end{aligned}$$

where $H^2_\Delta(\mathbb{R}^3 \backslash \Gamma^\varepsilon) := \left\{ u \in L^2(\mathbb{R}^3) : \Delta_{\mathbb{R}^3 \backslash \Gamma^\varepsilon} \in L^2(\mathbb{R}^3) \right\}$.

Let us remark that the generalized eigenfunctions of $H_\omega(\varepsilon)$ always satisfy the Sommerfeld radiation conditions.

Such a linear operator $-H_\omega(\varepsilon)$ is bounded from below and self-adjoint.
Its quadratic form $F_\omega(\varepsilon)$ has domain $H^1(\mathbb{R}^3)$ and acts as

$$F_\omega(\varepsilon)(u) = \|\nabla u\|^2_{L^2(\mathbb{R}^3)} + \left(\varepsilon^{-2} - 1\right) \langle DN_\omega(\varepsilon)\gamma_0(\varepsilon)u, \gamma_0(\varepsilon)u\rangle_{-\frac{1}{2},\frac{1}{2}} \ .$$

$-H_\omega(\varepsilon)$ rewrites as a Schrödinger operator with a singular delta-type perturbation

$$-H_\omega(\varepsilon)u = -\Delta u \dot{+} \left(\varepsilon^{-2} - 1\right) (DN_\omega(\varepsilon)\gamma_0(\varepsilon)u)\delta_{\Gamma^\varepsilon} \ .$$

Using boundary layer operators, one gets the Krein-type resolvent formula

$$\big(-H_\omega(\varepsilon)-z^2\big)^{-1}=\big(-\Delta-z^2\big)^{-1}-G_z(\varepsilon)\Lambda_z^\omega(\varepsilon)G_{-\bar z}(\varepsilon)^*\,,$$

where

$$\Lambda_z^\omega(\varepsilon):=\varepsilon\left(\frac{\varepsilon^2}{1-\varepsilon^2}+DN_{\varepsilon\omega}S_{\varepsilon z}\right)^{-1}DN_{\varepsilon\omega},$$

DN_z the Dirichlet-to-Neumann operator of Ω, S_z the boundary layer operators of Γ

$$S_z:=\gamma_0 SL_z,\quad SL_z\varphi(x):=\frac{1}{4\pi}\int_\Gamma\frac{e^{iz|x-y|}}{|x-y|}\,\varphi(y)\,d\sigma(y)$$

and

$$G_z(\varepsilon):=\varepsilon^{1/2}U_\varepsilon SL_{\varepsilon z},$$

U_ε the unitary operator in $L^2(\mathbb{R}^3)$

$$U_\varepsilon f(x):=\varepsilon^{-3/2}f(\varepsilon^{-1}(x-y_0)+y_0).$$

Resolvent expansions

Now, we are interested in the behavior of the resolvent

$$\big(-H_\omega(\varepsilon)-z^2\big)^{-1}: L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3),\quad \varepsilon\ll 1.$$

By the previous resolvent formula, this means that we need to study the behavior of

$$G_z(\varepsilon): H^{-1/2}(\Gamma)\to L^2(\mathbb{R}^3),\quad \varepsilon\ll 1$$

and

$$\Lambda_z^\omega(\varepsilon): H^{1/2}(\Gamma)\to H^{-1/2}(\Gamma),\quad \varepsilon\ll 1.$$

As regards $G_z(\varepsilon)$, one gets

$$\|G_z(\varepsilon)-G_z\|_{H^{-1/2}(\Gamma),L^2(\mathbb{R}^3)}\lesssim \varepsilon^{1/2},$$

$$G_z\phi(x):=\frac{1}{4\pi}\left(\int_\Gamma \phi(y)\,d\sigma(y)\right)\frac{e^{iz|x-y_0|}}{|x-y_0|},\quad \phi\in L^2(\Gamma)\,.$$

The study of the behavior of $\Lambda_z^{\omega}(\varepsilon)$ is much more involved.
At first, one takes into account the factorization

$$DN_{\varepsilon\omega} = S_{\varepsilon\omega}^{-1}\left(\frac{1}{2} + K_{\varepsilon\omega}\right),$$

where K_z is the Poincaré-Neumann operator

$$K_z := \gamma_0 DL_z, \quad DL_z\varphi(x) := \frac{1}{4\pi}\int_{\Gamma}\frac{\partial}{\partial\nu}\frac{e^{iz|x-y|}}{|x-y|}\,\varphi(y)\,d\sigma(y).$$

This leads to the rewriting

$$\Lambda_z^{\omega}(\varepsilon) = \varepsilon(1-\varepsilon^2)S_{\varepsilon\omega}^{-1}\left(\varepsilon^2 + (1-\varepsilon^2)\left(\frac{1}{2} + K_{\varepsilon\omega}\right)S_{\varepsilon z}S_{\varepsilon\omega}^{-1}\right)^{-1} S_{\omega\varepsilon}DN_{\omega\varepsilon}\,.$$

One has

$$S_z = S_0 + \sum_{n=1}^{+\infty} S_{(n)}\, z^n\,,$$

where the series converges in the Hilbert-Schmidt norm of operators on $H^{-1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$ and

$$S_{(n)}\phi(x) = \frac{1}{4\pi}\,\frac{i^n}{n!}\,\int_\Gamma |x-y|^{n-1}\phi(y)\,d\sigma(y)\,.$$

Moreover, $S_0 = \gamma_0 SL_0$ is coercive and thus $S_0^{-1} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is bounded and induces an inner product in $H^{1/2}(\Gamma)$:

$$\langle \phi, \varphi \rangle_{S_0^{-1}} := \int_\Gamma S_0^{-1}\phi(x)\varphi(x)\,d\sigma(x)\,.$$

As regards the Poincaré-Neumann operator, one has

$$K_z = K_0 + K_{(2)}\, z^2 + \sum_{n=3}^{+\infty} K_{(n)}\, z^n\,,$$

where the series converges in the Hilbert-Schmidt norm of operators in $H^{1/2}(\Gamma)$ and

$$K_{(n)}\phi(x) = -(n-1)\,\frac{1}{4\pi}\,\frac{i^n}{n!}\int_\Gamma \nu(y)\cdot(x-y)|x-y|^{n-3}\phi(y)\,d\sigma(y)\,.$$

Moreover, K_0 is a compact operator, $\sigma(K_0) \subseteq [-1/2, 1/2)$ and $\lambda_0 = -1/2$ is a simple eigenvalue with eigenfunction $\phi_0 = 1$.

Denoting by

$$P_0 : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$$

be the orthogonal (w.r.t. the inner product induced by S_0^{-1}) projector onto the subspace generated by the eigenfunction $\phi_0 = 1$, and defining

$$P_0^{\perp} := 1 - P_0$$

the projector onto the orthogonal,

$$K_0 : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$$

has the decomposition

$$K_0 = -\frac{1}{2}\, P_0 + P_0^{\perp} K_0 P_0^{\perp} \,.$$

This gives

$$\begin{aligned}\frac{1}{2} + K_{\varepsilon\omega} =& P_0\left((\varepsilon\omega)^2 K_{(2)} + (\varepsilon\omega)^3 K_{(3)} + O((\varepsilon\omega)^4)\right) P_0 \\ &+ P_0^{\perp}\left(\frac{1}{2} + K_0 + O((\varepsilon\omega)^2)\right) P_0^{\perp} \\ &+ P_0 O((\varepsilon\omega)^2) P_0^{\perp} + P_0^{\perp} O((\varepsilon\omega)^2) P_0 \,,\end{aligned}$$

and then

$$\begin{aligned}&\varepsilon^2 + \left(1-\varepsilon^2\right)\left(\frac{1}{2} + K_{\varepsilon\omega}\right) S_{\varepsilon z} S_{\varepsilon\omega}^{-1} \\ =& P_0\left(\varepsilon^2\left(1 + \omega^2 K_{(2)}\right) + \varepsilon^3\left(\omega^2 K_{(2)}\left(z-\omega\right) S_{(1)} S_0^{-1} + \omega^3 K_{(3)}\right) + O(\varepsilon^4)\right) P_0 \\ &+ P_0^{\perp}\left(\frac{1}{2} + K_0 + O(\varepsilon^2)\right) P_0^{\perp} + P_0 O(\varepsilon^2) P_0^{\perp} + P_0^{\perp} O(\varepsilon^2) P_0 \,.\end{aligned}$$

Taking into account the relation

$$P_0\left(1+\omega^2 K_{(2)}\right)P_0=\left(1-\frac{\omega^2}{\omega_M^2}\right)P_0\,,$$

$$\omega_M:=\sqrt{\frac{c_\Omega}{|\Omega|}}\,,\qquad c_\Omega:=\langle 1,1\rangle_{S_0^{-1}}=\int_\Gamma (S_0^{-1}1)(x)\,d\sigma(x)\,,$$

by the Schur complement one gets

$$\begin{aligned}\omega\neq\omega_M \quad\Rightarrow\quad &\varepsilon^2\left(\varepsilon^2+\left(1-\varepsilon^2\right)(1/2+K_{\varepsilon\omega})\,S_{\varepsilon z}S_{\varepsilon\omega}^{-1}\right)^{-1}\\ &=P_0\left(\frac{1}{1-\frac{\omega^2}{\omega_M^2}}+O(\varepsilon)\right)P_0+O(\varepsilon^2)\,,\end{aligned}$$

$$\begin{aligned}\omega=\omega_M \quad\Rightarrow\quad &\varepsilon^3\left(\varepsilon^2+\left(1-\varepsilon^2\right)(1/2+K_{\varepsilon\omega})\,S_{\varepsilon z}S_{\varepsilon\omega}^{-1}\right)^{-1}\\ &=P_0\left(\frac{4\pi}{c_\Omega}\,\frac{i}{z}+O(\varepsilon)\right)P_0+O(\varepsilon^2)\,.\end{aligned}$$

Combining these relations with

$$S_{\varepsilon\omega} DN_{\varepsilon\omega} = P_0^{\perp}(1/2 + K_0)P_0^{\perp} + \varepsilon^2\omega^2 K_{(2)} + O\left(\varepsilon^3\right) ,$$

finally one obtains

Theorem

1)

$$\textit{If } \omega \neq \omega_M, \quad \omega_M := \sqrt{\frac{c_\Omega}{|\Omega|}}, \quad c_\Omega := \int_\Gamma (S_0^{-1}1)(x)\, d\sigma(x), \ \textit{then}$$

$$\left\| \Lambda_z^\omega(\varepsilon) - \varepsilon\omega^2 \left(1 - \frac{\omega^2}{\omega_M^2}\right)^{-1} S_0^{-1} P_0 K_{(2)} \right\|_{H^{1/2}(\Gamma), H^{-1/2}(\Gamma)} \lesssim \varepsilon^2;$$

2) if $\omega = \omega_M$ *then*

$$\left\| \Lambda_z^\omega(\varepsilon) - \frac{4\pi i}{z|\Omega|} S_0^{-1} P_0 K_{(2)} \right\|_{H^{1/2}(\Gamma), H^{-1/2}(\Gamma)} \lesssim \varepsilon.$$

Inserting these estimates into the resolvent formula, one gets the following

Theorem

For any $z \in \mathbb{C}_+ \backslash i\mathbb{R}_+$ *and for any* $\varepsilon > 0$ *sufficiently small, one has*

$$\omega \neq \omega_M \;\Rightarrow\; \left\| \left(-H_\omega(\varepsilon) - z^2\right)^{-1} - \left(-\Delta - z^2\right)^{-1} \right\|_{L^2(\mathbb{R}^3), L^2(\mathbb{R}^3)} \lesssim \varepsilon \,,$$
$$\omega = \omega_M \;\Rightarrow\; \left\| \left(-H_\omega(\varepsilon) - z^2\right)^{-1} - R_z^{y_0} \right\|_{L^2(\mathbb{R}^3), L^2(\mathbb{R}^3)} \lesssim \varepsilon^{1/2},$$

where the bounded operator $R_z^{y_0}$ *has the kernel*

$$R_z^{y_0}(x,y) = \frac{1}{4\pi} \left(\frac{e^{iz|x-y|}}{|x-y|} - \frac{1}{iz}\, \frac{e^{iz|x-y_0|}}{|x-y_0|}\, \frac{e^{iz|y-y_0|}}{|y-y_0|} \right).$$

It turns out that $R_z^{y_0}$ is the resolvent of the self-adjoint operator Δ_{y_0} defined by

$$\mathrm{dom}(\Delta_{y_0}) := \Big\{ u \in L^2(\mathbb{R}^3) : u(x) = u_0(x) + \frac{q}{4\pi|x-y_0|}\,,\\ u_0 \in \dot{H}^2(\mathbb{R}^3),\ u_0(y_0) = 0,\ q \in \mathbb{C} \Big\}\,,$$

$$\Delta_{y_0} : \mathrm{dom}(\Delta_{y_0}) \subset L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)\,, \qquad \Delta_{y_0} u := \Delta u_0 \equiv \Delta u + q\,\delta_{y_0}\,.$$

By a limiting absorption principle, the previous resolvent estimates extend to the absolutely continuous spectrum and hence allow to control the behavior of the scattering solutions as $\varepsilon \ll 1$.

Let $\omega > 0$, $\alpha > 1/2$, and let $L^2_{-\alpha}(\mathbb{R}^3)$, $H^2_{-\alpha}(\mathbb{R}^3)$ denote be the weighted L^2 and Sobolev spaces with weight $(1+|x|^2)^{-\alpha/2}$.

Let $u^{inc}_\omega \in H^2_{-\alpha}(\mathbb{R}^3)$ be a solution of the homogeneous Helmholtz equation

$$(\Delta + \omega^2)u^{inc}_\omega = 0$$

and define

$$u^{scatt}_\omega := u_\omega - u^{inc}_\omega,$$

where u_ω is the unique solution of the s.b.v.p. corresponding to the incoming wave u^{inc}_ω.

Theorem

For any $\varepsilon > 0$ sufficiently small, one has, uniformly with respect to the choice of u_ω^{inc},

$$\left(u_\omega^{scatt}(\varepsilon)\right)(x) = \begin{cases} \varepsilon \, \frac{\omega^2 \, c_\Omega}{\omega_M^2 - \omega^2} \; u_\omega^{inc}(y_0) \, \frac{e^{i\omega|x-y_0|}}{4\pi|x-y_0|} + \left(r_\omega(\varepsilon)\right)(x) \,, & \omega \neq \omega_M \\ \\ \frac{i}{\omega} \; u_\omega^{inc}(y_0) \, \frac{e^{i\omega|x-y_0|}}{|x-y_0|} + \left(r_\omega(\varepsilon)\right)(x) \,, & \omega = \omega_M \end{cases}$$

$$\|r_\omega(\varepsilon)\|_{L^2_{-\alpha}(\mathbb{R}^3)} \lesssim \begin{cases} \varepsilon^{3/2} \,, & \omega \neq \omega_M \\ \varepsilon^{1/2} \,, & \omega = \omega_M \,. \end{cases}$$

Furthermore, with some more work, one also gets an expansions by varying both ω and ε:

Theorem

Let $\mathcal{I}_M$ be a real bounded interval containing ω_M and let $c_0 > 0$. Then for any $\varepsilon > 0$ sufficiently small, the expansion

$$\big(u_\omega^{\rm scatt}(\varepsilon)\big)(x) = \frac{\varepsilon\,\omega^2 c_\Omega\, u_\omega^{inc}(y_0)}{4\pi(\omega_M^2-\omega^2) - i\varepsilon\omega^3 c_\Omega}\,\frac{e^{i\omega|x-y_0|}}{|x-y_0|} + (r_\omega(\varepsilon))(x)\,,$$

$$\|r_\omega(\varepsilon)\|_{L^2_{-\alpha}(\mathbb{R}^3)} \lesssim \frac{\varepsilon^{3/2}}{\omega_M^2-\omega^2}\,, \qquad \alpha > 1/2\,.$$

holds uniformly with respect to ω in $\{\omega \in \mathcal{I}_M : |\omega-\omega_M| \geq c_0\,\varepsilon\}$ and u_ω^{inc} in any bounded subset of $H^2_{-\alpha}(\mathbb{R}^3)$.

Conclusions

• These estimates show that when ω approaches ω_M, the scattering system undergoes a transition between an asymptotically trivial scattering and a non-trivial one.

• The estimates hold in the whole space and improve all previously known expansion formulae which were limited to the far-field zone $|x| \gg 1$, without an uniform control-in-space of the errors.

• One gets an explicit self-adjoint operator governing the limiting scattering process; there was no prior knowledge about it.

THANKS FOR YOUR ATTENTION