On the inverse problem for Love waves in a layered, elastic half-space

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M. V. de Hoop, J. Garnier, A. lantchenko, J. R., Inverse problem for Love waves in a layered, elastic half-space, e-print (2023), https://hal.science/hal-03994654

Julien Ricaud

Inverse problem for Love waves

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Motivations

Imaging crustal and upper mantle structures: recovering the parameters of the medium from the dispersion curves of the surface waves.

Motivations

Imaging crustal and upper mantle structures: recovering the parameters of the medium from the dispersion curves of the surface waves.

Visual animations of surface waves [1]:

Love waves

Rayleigh waves

[1] Source: Seismological Facility for the Advancement of Geoscience (https://www.iris.edu/hq/inclass/animation/seismic_wave_motions4_waves_animated)

Motivations	Derivation	Characterization of Love waves	Recovering medium's parameters
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Motivations

Imaging crustal and upper mantle structures: recovering the parameters of the medium from the dispersion curves of the surface waves.





[BB-H96] Buchen & Ben-Hador. Free-mode surface-wave computations (1996). Geophys. J. Int. [Tho50] Thomson. Transmission of elastic avers through a stratified solid medium (1950). J. Appl. Phys. [Ha53] Haskell. The dispersion of surface waves on multilayered media (1953). Bull. Seismol. Soc. Am.

Motivations	Derivation	Characterization of Love waves	Recovering medium's parameters
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Derivation of the equation from the elastic wave equation in $\mathbb{R} \times \mathbb{R}^2 \times [0, +\infty)$, w/o source term.

- $\begin{cases} \rho \partial_{tt} \mathbf{u} = \operatorname{div} \boldsymbol{\tau}(\mathbf{u}) & \text{on } \mathbb{R} \times \mathbb{R}^2 \times [0, +\infty) \,, \qquad \text{(linear elastic wave equation)} \\ \boldsymbol{\tau}(\mathbf{u}) \cdot \mathbf{e}_3 = 0 & \text{at } z = 0 \,. \qquad \text{(stress-free (Neumann) BC)} \end{cases}$
- displacement vectors $\mathbf{u}(t, \mathbf{x}, z) = (u_1(t, \mathbf{x}, z), u_2(t, \mathbf{x}, z), u_3(t, \mathbf{x}, z));$
- mass density ρ(t, x, z);
- Cauchy stress tensor $au(\mathbf{u})$

Motivations	Derivation	Characterization of Love waves	Recovering medium's parameters
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Derivation of the equation from the elastic wave equation in $\mathbb{R} \times \mathbb{R}^2 \times [0, +\infty)$, w/o source term.

- $\begin{cases} \rho \partial_{tt} \mathbf{u} = \mathsf{div} \, \boldsymbol{\tau}(\mathbf{u}) & \text{on } \mathbb{R} \times \mathbb{R}^2 \times [0, +\infty) \,, \qquad (\text{linear elastic wave equation}) \\ \boldsymbol{\tau}(\mathbf{u}) \cdot \mathbf{e}_3 = 0 & \text{at } z = 0 \,. \qquad (\text{stress-free (Neumann) BC}) \end{cases}$
- displacement vectors $\mathbf{u}(t, \mathbf{x}, z) = (u_1(t, \mathbf{x}, z), u_2(t, \mathbf{x}, z), u_3(t, \mathbf{x}, z));$
- mass density $\rho(t, \mathbf{x}, z)$;
- Cauchy stress tensor $\boldsymbol{\tau}(\mathbf{u})$, given by Hookes' law $\boldsymbol{\tau}(\mathbf{u}) = \mathbf{C} \boldsymbol{\varepsilon}(\mathbf{u})$:
 - stiffness tensor C(t, x, z);
 - ▶ infinitesimal strain tensor ε , given by the strain-displacement equation

$$\boldsymbol{\varepsilon}(\mathbf{u}) = rac{\boldsymbol{\nabla}\mathbf{u} + \boldsymbol{\nabla}\mathbf{u}^{\mathsf{T}}}{2} \quad \Leftrightarrow \quad \varepsilon_{k\ell}(\mathbf{u}) = rac{\partial_{x_k} u_\ell + \partial_{x_\ell} u_k}{2} \,.$$

Motivations	Derivation	Characterization of Love waves	Recovering
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Derivation of the equation Assumptions

• symmetries $C_{ijk\ell} = C_{jik\ell} = C_{k\ell ij}$ (standard, physical assumption).

 $\begin{cases} \rho \partial_{tt} \mathbf{u} = \operatorname{div}(\mathbf{C}\boldsymbol{\varepsilon}(\mathbf{u})) \,, \\ \left. \left(\mathbf{C}\boldsymbol{\varepsilon}(\mathbf{u}) \right) \cdot \mathbf{e}_3 \right|_{\mathbf{z}=\mathbf{0}} = \mathbf{0} \,. \end{cases}$

Motivations O	Derivation O●	Characterization of Love waves 00000	Re	covering medium's parameters
Derivation of th Assumptions	ne equation		{ ($\left. \begin{aligned} \partial \partial_{tt} \mathbf{u} &= \operatorname{div}(\mathbf{C}\boldsymbol{\varepsilon}(\mathbf{u})) , \\ \mathbf{C}\boldsymbol{\varepsilon}(\mathbf{u})) \cdot \mathbf{e}_{3} \right _{z=0} &= 0 . \end{aligned}$
• symmetries C _{ij}	$k_{\ell} = C_{jik\ell} = C_{k\ell ij}.$		0	x
 stratified, home 	ogeneous in (x, y) -p	lane, time-independent mediur	n.	ρ ₁ , C ₁
				ρ ₂ , C ₂
				ρ ₃ , C ₃
				:
				·
				ρ_{n-1}, C_{n-1}
				ρ _n , C _n
				$ ho_\infty$, C $_\infty$
			.	z



Motivations O	Derivation O●	Characterization of Love waves 00000	Recovering medium's parameters 0000
Derivation of the Assumptions	e equation		$\begin{cases} \rho \partial_{tt} \mathbf{u} = \operatorname{div}(\mathbf{C}\boldsymbol{\varepsilon}(\mathbf{u})) , \\ \left(\mathbf{C}\boldsymbol{\varepsilon}(\mathbf{u})\right) \cdot \mathbf{e}_{3}\big _{z=0} = 0 . \end{cases}$
• symmetries <i>C</i> _{ijkℓ}	$= C_{jik\ell} = C_{k\ell ij}.$		0 x
 stratified, homog 	geneous in (x, y) -pl	ane, time-independent medium	ρ_1, λ_1, μ_1
• isotropic media:	$C_{ijk\ell} = \lambda \delta_i^j \delta_k^\ell + \mu ($	$\left(\delta_i^k \delta_i^\ell + \delta_i^\ell \delta_i^k\right)$	ρ_2, λ_2, μ_2
with $\lambda\equiv\lambda(z)$ a	nd $\mu \equiv \mu(z)$ the La	mé parameters.	ρ_3, λ_3, μ_3
			:
		-	$\rho_{n-1}, \lambda_{n-1}, \mu_{n-1}$
		-	ρ_n, λ_n, μ_n
			$ ho_{\infty}, \lambda_{\infty}, \mu_{\infty}$

z

Motivations O	Derivation O	Characterization of Love waves	Recove 0000	ring medium's parameters
Derivation Assumption	of the equation		$\begin{cases} \rho \partial_{tt} \\ (\mathbf{C}\boldsymbol{\varepsilon} \end{cases}$	$\begin{split} \mathbf{u} &= \operatorname{div}(\mathbf{C}\boldsymbol{\varepsilon}(\mathbf{u})) , \\ (\mathbf{u})) \cdot \mathbf{e}_3 \big _{z=0} &= 0 . \end{split}$
• symmetri	ies $C_{ijk\ell} = C_{jik\ell} = C$	- klij	0	x
 stratified 	, homogeneous in ((x, y)-plane, time-independent me	edium.	ρ_1, λ_1, μ_1
 isotropic 	media: $C_{ijk\ell} = \lambda \delta_i^j$	$\delta_k^\ell + \mu \left(\delta_i^k \delta_j^\ell + \delta_i^\ell \delta_j^k ight).$		ρ_2, λ_2, μ_2
Defining	the (<i>t</i> . x)-Fourier tr	ansform		$\rho_{3}, \lambda_{3}, \mu_{3}$
Ĺ	$\hat{u}_i(z) \equiv \hat{u}_i(\boldsymbol{\xi}, z, \omega)$:	$= \int_{\mathbb{R}^2} \int_{\mathbb{R}} u_i(\mathbf{x}, z, t) e^{\mathbf{i}\omega t} e^{\mathbf{i}\boldsymbol{\xi}\cdot\mathbf{x}} \mathrm{d}t \mathrm{d}\mathbf{x}$:
Then, $ ho \partial_t$	$\mathbf{u}_{t}\mathbf{u} = div(\mathbf{C}oldsymbol{arepsilon}(\mathbf{u}))$ re	ads		$\rho_{n-1}, \lambda_{n-1}, \mu_{n-1}$
$\begin{pmatrix} (\lambda + \mu)\xi_1^2 + \mu \boldsymbol{\xi} ^2 - \\ (\lambda + \mu)\xi \end{pmatrix}$	$ \begin{pmatrix} \partial_z \mu \partial_z + \rho \omega^2 \end{pmatrix} \qquad (\lambda + \mu)\xi_2^2 + \mu \boldsymbol{\xi} $	$ - \mu \xi_1 \xi_2 \qquad -i \left(\lambda \partial_z + \partial_z \mu \right) \xi_1 \\ ^2 - \left(\partial_z \mu \partial_z + \rho \omega^2 \right) \qquad -i \left(\lambda \partial_z + \partial_z \mu \right) \xi_2 \\ (\lambda \partial_z + \partial_z \mu) \xi_2 \qquad (\lambda \partial_z + \partial_z \mu) \xi_2 $	$\begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	ρ_n, λ_n, μ_n
$-1(\mu\sigma_z + \sigma_z)$	$(z\lambda) \xi_1 = -i (\mu c)$	$\mu_{z} + \sigma_{z}\lambda)\xi_{2} \qquad \mu_{z}\xi^{-} - \left(\sigma_{z}\left(\lambda + 2\mu\right)\sigma_{z} + \rho\omega^{-}\right)/\left(h\right)$	u ₃) (0)	$ ho_\infty$, λ_∞ , μ_∞
and $(\mathbf{C} \mathbf{arepsilon})$	$\left u \right)) \cdot \mathbf{e}_{3} \big _{z=0} = 0$ rea	ds		
	($i\xi_1\hat{u}_3(0) + \partial_z\hat{u}_1(0) = 0$,	7	
	{	$i\xi_2\hat{u}_3(0) + \partial_z\hat{u}_2(0) = 0$,	¥ -	
	$i\lambda(0)(\xi_1\hat{u}_1(0) + \xi_2\hat{u}_2)$	$(0)) + (\lambda(0) + 2\mu(0)) \partial_z \hat{u}_3(0) = 0.$		

Motivations O	Derivation O	Characterization of Love waves 00000	Recovering medium's paramete	rs
Derivation Assumption	of the equation		$\begin{cases} \rho \partial_{tt} \mathbf{u} = div(\mathbf{C}\boldsymbol{\varepsilon}(\mathbf{u})) ,\\ \left(\mathbf{C}\boldsymbol{\varepsilon}(\mathbf{u})\right) \cdot \mathbf{e}_{3} \big _{z=0} = 0 \end{cases}$	
 symmetrie 	es $C_{ijk\ell} = C_{jik\ell} = C_{k\ell ij}$.		0	x
 stratified, 	homogeneous in (x, y)	y)-plane, time-independent me	redium. ρ_1, λ_1, μ_1	•
 isotropic 	media: $C_{ijk\ell} = \lambda \delta^j_i \delta^\ell_k$ +	$-\mu\left(\delta_i^k\delta_j^\ell+\delta_i^\ell\delta_j^k ight).$	ρ_2, λ_2, μ_2	
Defining t	he (t, \mathbf{x}) -Fourier transf	form	$\rho_{3}, \lambda_{3}, \mu_{3}$	
û	$\hat{u}_i(z) \equiv \hat{u}_i(\boldsymbol{\xi}, z, \omega) := \int_{\mathbb{R}}$	$\int_{\mathbb{R}^2} \int_{\mathbb{R}} u_i(\mathbf{x}, z, t) e^{\mathbf{i}\omega t} e^{\mathbf{i}\boldsymbol{\xi}\cdot\mathbf{x}} \mathrm{d}t \mathrm{d}\mathbf{x}$:	
and (ϕ_1,ϕ_2)	$(a_2, \phi_3)^{T} := P(\boldsymbol{\xi})(\hat{u}_1, \hat{u}_2, \hat{u}_2, \hat{u}_2)$	$(\hat{u}_3)^{T}$, with $P(\boldsymbol{\xi}) := \begin{pmatrix} \xi_2/ \boldsymbol{\xi} & -\xi_1/ \\ \xi_1/ \boldsymbol{\xi} & \xi_2/ \boldsymbol{\xi} \\ 0 & 0 \end{pmatrix}$	$\begin{vmatrix} \boldsymbol{\xi} & 0 \\ \boldsymbol{\xi} & 0 \\ 1 \end{vmatrix}$	
Then, $ ho\partial_t$	$\mathbf{u} = div(\mathbf{C}\boldsymbol{\varepsilon}(\mathbf{u}))$ reads	X	$\rho_{n-1}, \lambda_{n-1}, \mu_{n-1}$	L
$\begin{pmatrix} (\lambda + \mu)\xi_1^2 + \mu \boldsymbol{\xi} ^2 - (\lambda + \mu)\xi_1 \\ (\lambda + \mu)\xi_1 \\ -i(\mu\partial_z + \partial_z \end{pmatrix}$	$ \begin{array}{l} \left(\partial_{z} \mu \partial_{z} + \rho \omega^{2} \right) & \left(\lambda + \mu \right) \xi_{1} \xi_{1} \xi_{2} \\ \xi_{2} & \left(\lambda + \mu \right) \xi_{2}^{2} + \mu \boldsymbol{\xi} ^{2} - \left(\boldsymbol{\ell} \cdot \boldsymbol{\ell} \right) \\ \lambda \xi_{1} & -\mathbf{i} \left(\mu \partial_{z} + \partial_{z} \right) \end{array} $	$ \begin{array}{l} \sum_{\lambda_{2}} & -\mathrm{i} \left(\lambda \partial_{x} + \partial_{z} \mu \right) \xi_{1} \\ \partial_{z} \mu \partial_{z} + \rho \omega^{2} \right) & -\mathrm{i} \left(\lambda \partial_{z} + \partial_{z} \mu \right) \xi_{2} \\ \lambda \rangle \xi_{2} & \mu \boldsymbol{\xi} ^{2} - \left(\partial_{z} \left(\lambda + 2 \mu \right) \partial_{z} + \rho \omega^{2} \right) \end{array} \right) \left(\end{array} $	$ \begin{pmatrix} \hat{u}_1\\ \hat{u}_2\\ \hat{u}_3 \end{pmatrix} = \begin{pmatrix} 0\\ 0\\ 0 \end{pmatrix} \qquad \qquad$	
			$ ho_\infty$, λ_∞ , μ_∞	
and $({f C}arepsilon({f u}))$	$\mathbf{u}))\cdot\mathbf{e}_{3}\big _{z=0}=0$ reads			
1	ſ	$i\xi_1\hat{u}_3(0) + \partial_z\hat{u}_1(0) = 0$,	7	
ł		$i\xi_2\hat{u}_3(0)+\partial_z\hat{u}_2(0)=0,$	¥ -	
l	$i\lambda(0) (\xi_1 \hat{u}_1(0) + \xi_2 \hat{u}_2(0))$	$+ (\lambda(0) + 2\mu(0)) \partial_z \hat{u}_3(0) = 0.$		

Motivations O	Derivation O●	Characterization of Love waves 00000	Recor	vering medium's parameters 90
Derivation Assumption	of the equation		$\begin{cases} \rho \partial \\ (\mathbf{C}) \end{cases}$	$\begin{split} _{tt} \mathbf{u} &= \operatorname{div}(\mathbf{C}\boldsymbol{\varepsilon}(\mathbf{u})) , \\ \mathbf{\varepsilon}(\mathbf{u})) \cdot \mathbf{e}_{3}\big _{z=0} = 0 . \end{split}$
• symmetrie	es $C_{ijk\ell} = C_{jik\ell} = C_{k\ell}$	ij.	0	x
 stratified, 	homogeneous in $(x,$	y)-plane, time-independent med	ium.	ρ_1, λ_1, μ_1
• isotropic	media: $C_{ijk\ell} = \lambda \delta_i^j \delta_k^\ell$	$+ \mu \left(\delta_i^k \delta_j^\ell + \delta_i^\ell \delta_j^k \right).$		ρ_2, λ_2, μ_2
Defining t	he (<i>t</i> . x)-Fourier tran	sform		$\rho_{3}, \lambda_{3}, \mu_{3}$
\hat{u}_i and (ϕ_1,ϕ_2)	$\hat{u}_i(\boldsymbol{z}) \equiv \hat{u}_i(\boldsymbol{\xi}, \boldsymbol{z}, \omega) := \hat{u}_i(\boldsymbol{\xi}, \boldsymbol{z}, \omega) := \hat{u}_i(\boldsymbol{\xi}, \boldsymbol{z}, \omega)^T := \boldsymbol{P}(\boldsymbol{\xi})(\hat{u}_1, \hat{u}_2)$	$\int_{\mathbb{R}^2} \int_{\mathbb{R}} u_i(\mathbf{x}, z, t) e^{i\omega t} e^{i\boldsymbol{\xi}\cdot\mathbf{x}} dt d\mathbf{x}$ $(\mathbf{x}, \hat{u}_3)^{T}, \text{ with } P(\boldsymbol{\xi}) := \begin{pmatrix} \xi_2/ \boldsymbol{\xi} & -\xi_1/ \boldsymbol{\xi} \\ \xi_1/ \boldsymbol{\xi} & \xi_2/ \boldsymbol{\xi} \\ & & & & & \\ \end{pmatrix}$	0 0 1	:
Then, $ ho\partial_{tt}$	$\mathbf{u} = div(\mathbf{C}\boldsymbol{\varepsilon}(\mathbf{u}))$ read	S		$\rho_{n-1}, \lambda_{n-1}, \mu_{n-1}$
$\begin{pmatrix} -\partial_{z}\mu\partial_{z} + \mu z \\ 0 \\ 0 \end{pmatrix}$	$\begin{aligned} \boldsymbol{\xi} ^2 - \rho \omega^2 & 0 \\ -\partial_z \mu \partial_z + (\lambda + 2\mu) \boldsymbol{\xi} \\ -\mathbf{i} \boldsymbol{\xi} (\mu \partial_z + \partial_z) \end{aligned}$	$ \begin{pmatrix} 0 \\ \rho \omega^2 & -i \boldsymbol{\xi} (\lambda \partial_x + \partial_x \mu) \\ \lambda & -\partial_z (\lambda + 2\mu) \partial_z + \mu \boldsymbol{\xi} ^2 - \rho \omega^2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ \phi_3 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ \phi_3 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ \phi_3 \end{pmatrix} =$		ρ_n, λ_n, μ_n
,	1210		()	$ ho_\infty$, λ_∞ , μ_∞
and $({f C}arepsilon({f u}$	$\left \mathbf{u} \right) \cdot \mathbf{e}_3 \Big _{z=0} = 0$ reads			
	$\begin{cases} \\ & i\lambda(0) \boldsymbol{\xi} \phi_2(0)+(\lambda) \end{cases}$	$\begin{split} \partial_z \phi_1(0) &= 0 , \\ \mathrm{i} \boldsymbol{\xi} \phi_3(0) + \partial_z \phi_2(0) &= 0 , \\ \Lambda(0) + 2\mu(0)) \partial_z \phi_3(0) &= 0 . \end{split}$	z	

Motivations	Derivation	Characterization of Love waves	Recovering medium's parameters
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Love waves: definition (in our context)

n+1 layers with constant shear modulus $\mu > 0$ and density $\rho > 0$. $H_1 = 0$





Motivations	Derivation	Characterization of Love waves	Recovering medium's parameters
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Love waves:	definition (in our context)		

Love waves: definition (in our context)

$$n + 1 \text{ layers with constant shear modulus } \mu > 0 \text{ and density } \rho > 0.$$

$$H_{1} = 0 \xrightarrow{0} \xrightarrow{\mathbf{x}} \rho_{1}, \mu_{1}$$

$$H_{2} \xrightarrow{\rho_{1}, \mu_{1}} \rho_{2}, \mu_{2}$$

$$H_{3} \xrightarrow{\rho_{2}, \mu_{2}} \rho_{3}, \mu_{3}$$

$$H_{4} \xrightarrow{0} \rho_{3}, \mu_{3}$$

Defining $\nu_j \equiv \nu_j(\omega, k) := \omega_1 \sqrt{k^2/\omega^2 - C_j^{-2}}$ (with Im $\nu_j \leq 0$), where $\mathcal{C}_i := \sqrt{\mu_j/
ho_j}$, then on each layer ϕ is either affine or of the form $A_{i,+}e^{+\nu_j z} + A_{i,-}e^{-\nu_j z}$.

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Love wa	ves: definition (in	our context)			
n+1 la A Love	ayers with constant wave exists at (ω, k) $u(\phi')' = u(\omega)^2 (\alpha/u - 1)$	shear modulus $\mu > 0$ and ϕ (c) : $\Rightarrow \exists \phi \equiv \phi_{\omega,k} \in L^2((0, -k^2/\omega^2))$ on $[0, +\infty)$	density $\rho > 0$. + ∞))\{0} s.t. $k/\omega > 1/0$	$H_1 = 0 \stackrel{0}{-} H_2 \stackrel{-}{-} H_3 \stackrel{-}{-} H_4 \stackrel{-}{-} $	$\begin{array}{c} & \mathbf{x} \\ \hline \rho_1, \ \mu_1 \\ \hline \rho_2, \ \mu_2 \\ \hline \rho_3, \ \mu_3 \end{array}$
$\begin{cases} \phi \in \mathcal{O} \\ \mu \phi' \in \mathcal{O} \end{cases}$	$\mathcal{L}([0, +\infty)) = \mu \omega (\mu) \mu$ $\mathcal{L}([0, +\infty)) \text{ with } \lim_{+\infty} \varphi$ $\in \mathcal{L}([0, +\infty)) \text{ with } \varphi$	$a \phi = 0,$ b'(0) = 0.	K/ W > 1/ X	-∞ , H _{n+1} —	:
Definin	g $ u_j \equiv u_j(\omega, k) := \omega$	$\sqrt{k^2/\omega^2-C_j^{-2}}$ (with Im $ u$	$v_j \leqslant 0$), where	H _n —	$\frac{\rho_{n-1}, \mu_{n-1}}{\rho_{n}, \mu_{n}}$
$egin{array}{lll} C_j := & \ A_{j,+} e^+ \ k/\omega > \end{array}$	$\sqrt{\mu_j/ ho_j}$, then on each $\sqrt{\mu_j/ ho_j}$, then on each $2^{ u_j z} + A_{j,-} e^{- u_j z}$. The $1/C_{\infty} \ge 0$, and A_{∞} ,	h layer ϕ is either affine or us, $\phi \in L^2((0, +\infty)) \Rightarrow \nu_{\infty}$ $_+ = 0 \Rightarrow \lim_{+\infty} \phi = 0.$	of the form $>$ 0, i.e.,	n_{n+1} —	$\rho_{\infty}, \mu_{\infty}$
				(<i>H</i> ∞ :	$z = +\infty$

Characterization of Love waves

Motivations	Derivation	Characterization of Love waves	Recovering medium's parameter
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Love waves: characterization

Assuming for simplicity that (ω, k) is s.t. $\nu_j \neq 0$, if $\phi \equiv \phi_{\omega,k}$ exists, then

$$\phi(z) = \begin{cases} 2\alpha_1 \cosh[\nu_1 z], & \text{if} \quad 0 \leq z < H_2, \\ \alpha_j e^{-\nu_j z} + \beta_j e^{+\nu_j z}, & \text{if} \quad H_j \leq z < H_{j+1}, \quad \forall j \in \llbracket 2, n \rrbracket, \\ \alpha_{n+1} e^{-\nu_{n+1} z}, & \text{if} \quad H_{n+1} \leq z < +\infty. \end{cases}$$

Love waves: characterization

Assuming for simplicity that (ω, k) is s.t. $\nu_j \neq 0$, if $\phi \equiv \phi_{\omega,k}$ exists, then

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The 2*n* continuity conditions (on ϕ and $\mu \phi'$) at the boundaries $\{H_j\}_{2 \leq j \leq n+1}$ yield

$$\begin{cases} 2\alpha_{1}\cosh[\nu_{1}H_{2}] = \beta_{2}e^{+\nu_{2}H_{2}} + \alpha_{2}e^{-\nu_{2}H_{2}}, \\ 2\mu_{1}\nu_{1}\alpha_{1}\sinh[\nu_{1}H_{2}] = \mu_{2}\nu_{2}\left(\beta_{2}e^{+\nu_{2}H_{2}} - \alpha_{2}e^{-\nu_{2}H_{2}}\right), \\ \beta_{j-1}e^{+\nu_{j-1}H_{j}} + \alpha_{j-1}e^{-\nu_{j-1}H_{j}} = \beta_{j}e^{+\nu_{j}H_{j}} + \alpha_{j}e^{-\nu_{j}H_{j}}, \qquad \forall j \in [\![3,n]\!], \\ \mu_{j-1}\nu_{j-1}\left(\beta_{j-1}e^{+\nu_{j-1}H_{j}} - \alpha_{j-1}e^{-\nu_{j-1}H_{j}}\right) = \mu_{j}\nu_{j}\left(\beta_{j}e^{+\nu_{j}H_{j}} - \alpha_{j}e^{-\nu_{j}H_{j}}\right), \qquad \forall j \in [\![3,n]\!], \\ \beta_{n}e^{+\nu_{n}H_{n+1}} + \alpha_{n}e^{-\nu_{n}H_{n+1}} = \alpha_{n+1}e^{-\nu_{n+1}H_{n+1}}, \\ \mu_{n}\nu_{n}\left(\beta_{n}e^{+\nu_{n}H_{n+1}} - \alpha_{n}e^{-\nu_{n}H_{n+1}}\right) = -\mu_{n+1}\nu_{n+1}\alpha_{n+1}e^{-\nu_{n+1}H_{n+1}}. \end{cases}$$

Love waves: characterization

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$$\phi(z) = \begin{cases} 2\alpha_1 \cosh[\nu_1 z], & \text{if} \quad 0 \leq z < H_2, \\ \alpha_j e^{-\nu_j z} + \beta_j e^{+\nu_j z}, & \text{if} \quad H_j \leq z < H_{j+1}, \quad \forall j \in \llbracket 2, n \rrbracket, \\ \alpha_{n+1} e^{-\nu_{n+1} z}, & \text{if} \quad H_{n+1} \leq z < +\infty. \end{cases}$$

The 2*n* continuity conditions (on ϕ and $\mu \phi'$) at the boundaries $\{H_j\}_{2 \leqslant j \leqslant n+1}$ yield

$$\begin{cases} 2\alpha_{1}\cosh[\nu_{1}H_{2}] = \beta_{2}e^{+\nu_{2}H_{2}} + \alpha_{2}e^{-\nu_{2}H_{2}}, \\ 2\mu_{1}\nu_{1}\alpha_{1}\sinh[\nu_{1}H_{2}] = \mu_{2}\nu_{2}\left(\beta_{2}e^{+\nu_{2}H_{2}} - \alpha_{2}e^{-\nu_{2}H_{2}}\right), \\ \beta_{j-1}e^{+\nu_{j-1}H_{j}} + \alpha_{j-1}e^{-\nu_{j-1}H_{j}} = \beta_{j}e^{+\nu_{j}H_{j}} + \alpha_{j}e^{-\nu_{j}H_{j}}, \qquad \forall j \in [\![3,n]\!], \\ \mu_{j-1}\nu_{j-1}\left(\beta_{j-1}e^{+\nu_{j-1}H_{j}} - \alpha_{j-1}e^{-\nu_{j-1}H_{j}}\right) = \mu_{j}\nu_{j}\left(\beta_{j}e^{+\nu_{j}H_{j}} - \alpha_{j}e^{-\nu_{j}H_{j}}\right), \qquad \forall j \in [\![3,n]\!], \\ \beta_{n}e^{+\nu_{n}H_{n+1}} + \alpha_{n}e^{-\nu_{n}H_{n+1}} = \alpha_{n+1}e^{-\nu_{n+1}H_{n+1}}, \\ \mu_{n}\nu_{n}\left(\beta_{n}e^{+\nu_{n}H_{n+1}} - \alpha_{n}e^{-\nu_{n}H_{n+1}}\right) = -\mu_{n+1}\nu_{n+1}\alpha_{n+1}e^{-\nu_{n+1}H_{n+1}}. \end{cases}$$

A Love wave exists if there exists a non-zero solution $(\alpha_1, \beta_1, \alpha_2, \beta_2, \dots, \alpha_{n+1}, \beta_{n+1})$.

$$\begin{array}{c|cccc} \underbrace{\text{Motivations}}_{0} & \underbrace{\text{Derivation}}_{0} & \underbrace{\text{Characterization of Low evaluations}}_{0000} & \underbrace{\text{Recovering medium's parameter}}_{0000} \\ \hline \\ \text{Love waves: characterization} \\ \text{I.e., if } D_n := \det \mathbb{M}_n = 0, \text{ for } \mathbb{M}_n := \begin{pmatrix} L_1^r & R_2 & \mathbb{O}_2 & \mathbb{O}_2 & \mathbb{O}_2 & 0 \\ 0 & L_2 & R_3 & \ddots & \mathbb{O}_2 & \vdots \\ 0 & 2 & \ddots & \ddots & \ddots & \mathbb{O}_2 & \vdots \\ 0 & 0 & 2 & \mathbb{O}_2 & \ddots & \mathbb{O}_2 & \vdots \\ 0 & 0 & 0 & 2 & \mathbb{O}_2 & \mathbb{O}_2 & L_n & R_{n+1}^I \end{pmatrix} \\ \text{where } L_1^r := 2 \begin{pmatrix} \cosh[\nu_1 H_2] \\ \mu_1 \nu_1 \sinh[\nu_1 H_2] \end{pmatrix}, R_{n+1}^{l} := \begin{pmatrix} -e^{-\nu_n + 1H_{n+1}} \\ +\mu_{n+1} \nu_{n+1}e^{-\nu_{n+1}H_{n+1}} \end{pmatrix}, \text{ and} \\ \forall j \ge 2, \begin{cases} L_j := \begin{pmatrix} +e^{-\nu_j H_j + 1} & +e^{+\nu_j H_{j+1}} \\ -\mu_j \nu_j e^{-\nu_j H_{j+1}} & +\mu_j \nu_j e^{+\nu_j H_{j+1}} \end{pmatrix}, \\ R_j := \begin{pmatrix} -e^{-\nu_j H_j} & -e^{+\nu_j H_j} \\ +\mu_j \nu_j e^{-\nu_j H_j} & -\mu_j \nu_j e^{+\nu_j H_j} \end{pmatrix}. \end{cases}$$

$$\begin{array}{c|cccc} \underbrace{\text{Motivations}}_{0} & \underbrace{\text{Derivation}}_{0} & \underbrace{\text{Characterization of Low eaves}}_{0000} & \underbrace{\text{Recovery medium's parameter}}_{0000} \\ \hline \end{array}$$
Love waves: characterization
$$I.e., \text{ if } D_n := \det \mathbb{M}_n = 0, \text{ for } \mathbb{M}_n := \begin{pmatrix} L_1^r & R_2 & \mathbb{O}_2 & \mathbb{O}_2 & \mathbb{O}_2 & 0 \\ 0 & L_2 & R_3 & \ddots & \mathbb{O}_2 & \vdots \\ 0 & 2 & \ddots & \ddots & \ddots & \mathbb{O}_2 & \vdots \\ 0 & 0 & 2 & \mathbb{O}_2 & \ddots & \mathbb{O}_2 & \vdots \\ 0 & 0 & 0 & 2 & \mathbb{O}_2 & \mathbb{O}_2 & L_n & R_{n+1}^I \end{pmatrix} \\ \text{where } L_1^r := 2 \begin{pmatrix} \cosh[\nu_1 H_2] \\ \mu_1 \nu_1 \sinh[\nu_1 H_2] \end{pmatrix}, R_{n+1}^r := \begin{pmatrix} -e^{-\nu_n + 1H_{n+1}} \\ +\mu_{n+1} \nu_{n+1}e^{-\nu_{n+1}H_{n+1}} \end{pmatrix}, \text{ and} \\ \forall j \ge 2, \begin{cases} L_j := \begin{pmatrix} +e^{-\nu_j H_{j+1}} & +e^{+\nu_j H_{j+1}} \\ -\mu_j \nu_j e^{-\nu_j H_{j+1}} & +\mu_j \nu_j e^{+\nu_j H_{j+1}} \end{pmatrix}, \\ R_j := \begin{pmatrix} -e^{-\nu_j H_j} & -e^{+\nu_j H_j} \\ +\mu_j \nu_j e^{-\nu_j H_j} & -\mu_j \nu_j e^{+\nu_j H_j} \end{pmatrix}. \end{cases}$$

→ We read from it that there is at most one solution: the submatrix $\tilde{\mathbb{M}}_n$ where the first row and the last column are removed, is block (upper) triangular with diagonal blocks $2\mu_1\nu_1 \sinh[\nu_1H_2]$ and the L_j 's, hence det $\tilde{\mathbb{M}}_n = 2^n \sinh[\nu_1H_2] \prod_{i=1}^n \mu_i \nu_j \neq 0$.

$$\begin{array}{c|ccccc} \underbrace{\text{Motivations}}_{0} & \underbrace{\text{Derivation}}_{0} & \underbrace{\text{Characterization of Low waves}}_{0000} & \underbrace{\text{Recovering medium's paramete}}_{0000} \\ \hline \\ \hline \\ \text{Love waves: characterization} \\ \text{I.e., if } D_n := \det \mathbb{M}_n = 0, \text{ for } \mathbb{M}_n := \begin{pmatrix} L_1' & R_2 & \mathbb{O}_2 & \mathbb{O}_2 & \mathbb{O}_2 & 0 \\ 0 & L_2 & R_3 & \ddots & \mathbb{O}_2 & \vdots \\ \vdots & \mathbb{O}_2 & \ddots & \ddots & \ddots & \mathbb{O}_2 & \vdots \\ \vdots & \mathbb{O}_2 & \ddots & \ddots & \ddots & \mathbb{O}_2 & \vdots \\ \vdots & \mathbb{O}_2 & 0 & \mathbb{O}_2 & \mathbb{O}_2 & L_n & R_{n+1}^{\prime} \end{pmatrix} \\ \text{where } L_1' := 2 \begin{pmatrix} \cosh[\nu_1 H_2] \\ \mu_1 \nu_1 \sinh[\nu_1 H_2] \end{pmatrix}, R_{n+1}' := \begin{pmatrix} -e^{-\nu_n + 1H_{n+1}} \\ +\mu_{n+1} \nu_{n+1} e^{-\nu_{n+1}H_{n+1}} \end{pmatrix}, \text{ and} \\ \forall j \ge 2, \begin{cases} L_j := \begin{pmatrix} +e^{-\nu_j H_{j+1}} & +e^{+\nu_j H_{j+1}} \\ -\mu_j \nu_j e^{-\nu_j H_{j+1}} & +\mu_j \nu_j e^{+\nu_j H_{j+1}} \end{pmatrix}, \\ R_j := \begin{pmatrix} -e^{-\nu_j H_j} & -e^{+\nu_j H_j} \\ +\mu_j \nu_j e^{-\nu_j H_j} & -\mu_j \nu_j e^{+\nu_j H_j} \end{pmatrix}. \end{cases}$$

→ We read from it that there is at most one solution: the submatrix M_n where the first row and the last column are removed, is block (upper) triangular with diagonal blocks $2\mu_1\nu_1 \sinh[\nu_1H_2]$ and the L_j 's, hence det $M_n = 2^n \sinh[\nu_1H_2] \prod_{j=1}^n \mu_j\nu_j \neq 0$. (It also holds if some ν_j 's are zero.)

Love waves: dispersion relation

Proposition: dispersion relation

Let $n \in \mathbb{N} \setminus \{0\}$ and $T_j := H_{j+1} - H_j$, $j \in \llbracket 1, n+1 \rrbracket$, be the layers' thickness. Then,

$$\exists \text{ Love wave at } (\omega, k) \Leftrightarrow \begin{cases} f_n(\omega, k) := \mu_{\infty} \nu_{\infty}(\omega, k) P_n(\omega, k) + Q_n(\omega, k) = 0, \\ k \neq \omega/C_{\infty}, \end{cases}$$

where the P_n 's and Q_n 's are defined recursively by $P_0 = 1$, $Q_0 = 0$, and

$$egin{pmatrix} P_m \ Q_m \end{pmatrix} = M_m egin{pmatrix} P_{m-1} \ Q_{m-1} \end{pmatrix} \quad ext{for all } m \in \llbracket 1, n
rbracket,$$

with

$$M_m := \begin{cases} \begin{pmatrix} \cosh[\nu_m T_m] & \sinh[\nu_m T_m]/(\mu_m \nu_m) \\ \mu_m \nu_m \sinh[\nu_m T_m] & \cosh[\nu_m T_m] \end{pmatrix} & \text{if } \nu_m \neq 0, \\ \begin{pmatrix} 1 & T_m/\mu_m \\ 0 & 1 \end{pmatrix} & \text{if } \nu_m = 0. \end{cases}$$

" $k \neq \omega/C_{\infty}$ " can be replaced by " $\omega/C_{\infty} < k < \omega/C_0$ " ($C_0 := \min_{[0,+\infty)} C$), because the solutions to $f_n(\omega, k) = 0$ are s.t. $k \in [\omega/C_{\infty}, \omega/C_0)$.

Motivations O	Derivation OO	Characterization of Love waves	Recovering medium's paramete

Love waves: numerics



1/C

1/C₁

1/C₃

1/C₀

1/C7

Definition-Lemma

Let $n \ge 1$. For any fixed $\omega > 0$, let the $k_{\ell}(\omega)$'s be the decreasingly ordered values $k \in \mathbb{R} \setminus \{\omega/C_{\infty}\}$ for which (ω, k) solves the dispersion relation $f_n(\omega, k) = 0$, i.e., a Love wave exists. Then, $k_{\ell} \in (\omega/C_{\infty}, \omega/C_0)$ and f_n is real-valued on $(0, +\infty) \times [\omega/C_{\infty}, \omega/C_0)$.



Motivations	Derivation	Characterization of Love waves	Recovering medium's parameters
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Theorem

Let $n \ge 1$. For any ℓ , there exists $\omega_{\ell} \ge 0$ s.t.

$$(\omega_{\ell}, +\infty)
ightarrow (1/\mathcal{C}_{\infty}, 1/\mathcal{C}_{0})$$

 $\omega \mapsto k_{\ell}(\omega)/\omega$

is analytic, bijective, increasing.



Motivations	Derivation	Characterization of Love waves	Recovering medium's parameters
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Theorem

Let $n \ge 1$. For any ℓ , there exists $\omega_{\ell} \ge 0$ s.t.

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 $\omega \mapsto k_{\ell}(\omega)/\omega$

is analytic, bijective, increasing.



 $\Rightarrow C_{\infty}$ and $C_0 := \min_i C_i$ are the inverse of the upper and lower limit values.

Motivations	Derivation	Characterization of Love waves	Recovering medium's parameters
0	00	00000	0000

Theorem

Let $n \ge 1$. For any ℓ , there exists $\omega_{\ell} \ge 0$ s.t.

$$(\omega_{\ell}, +\infty) \rightarrow (1/\mathcal{C}_{\infty}, 1/\mathcal{C}_{0})$$

 $\omega \mapsto k_{\ell}(\omega)/\omega$

is analytic, bijective, increasing.



Idea of proof:

 ω fixed: k_ℓ(ω)'s in finite number, hence isolated.Uses the simplicity of the Love waves and complex analysis (Identity theorem).

Motivations	Derivation	Characterization of Love waves	Recovering medium's parameters
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Theorem

Let $n \ge 1$. For any ℓ , there exists $\omega_{\ell} \ge 0$ s.t.

 $(\omega_{\ell}, +\infty) \rightarrow (1/C_{\infty}, 1/C_{0})$ $\omega \mapsto k_{\ell}(\omega)/\omega$

is analytic, bijective, increasing.



Idea of proof:

- ω fixed: $k_{\ell}(\omega)$'s in finite number, hence isolated.
- Branches ω → (k_ℓ(ω), φ_ℓ(ω)) exist on an open interval, with k_ℓ and φ_ℓ analytic. Uses by analytic perturbation theory (Kato–Rellich theorem), relying on the simplicity and isolation.

Motivations	Derivation	Characterization of Love waves	Recovering medium's parameters
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Theorem

Let $n \ge 1$. For any ℓ , there exists $\omega_{\ell} \ge 0$ s.t.

$$(\omega_{\ell}, +\infty) \to (1/C_{\infty}, 1/C_{0})$$

 $\omega \mapsto k_{\ell}(\omega)/\omega$

is analytic, bijective, increasing.



Idea of proof:

- ω fixed: $k_{\ell}(\omega)$'s in finite number, hence isolated.
- Branches $\omega \mapsto (k_{\ell}(\omega), \phi_{\ell}(\omega))$ exist on an open interval, with k_{ℓ} and ϕ_{ℓ} analytic.
- Increasing: direct computation. Since $\omega \mapsto k_{\ell}(\omega)/\omega$ is differentiable.

Motivations O	Derivation OO	Characterization of Love waves	Recovering medium's parameters
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Definition: number of branches $k_{\ell}(\omega)/\omega$ above or equal to y

Let $n \ge 1$, $\omega > 0$ and $y \in (1/C_{\infty}, 1/C_0)$. Define

$$\mathsf{N}(\omega, y) := \# \left\{ \ell \ge 1 : \frac{k_{\ell}(\omega)}{\omega} \ge y \right\}$$

with $k_{\ell}(\omega) = -\infty$ if k_{ℓ} is undefined at ω .



Motivations	Derivation	Characterization of Love waves	Recovering medium's parameters
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Definition: number of branches $k_{\ell}(\omega)/\omega$ above or equal to y

Let $n \ge 1$, $\omega > 0$ and $y \in (1/C_{\infty}, 1/C_{0})$. Define

$$\mathcal{N}(\omega,y) := \# \left\{ \ell \geqslant 1 \, : \, rac{k_\ell(\omega)}{\omega} \geqslant y
ight\} \, = \, \max \left\{ \ell \geqslant 1 \, : \, rac{k_\ell(\omega)}{\omega} \geqslant y > rac{k_{\ell+1}(\omega)}{\omega}
ight\},$$

with $k_{\ell}(\omega) = -\infty$ if k_{ℓ} is undefined at ω .

Note: $\omega \mapsto \mathcal{N}(\omega, y)$ is nondecreasing for $y \in (1/\mathcal{C}_{\infty}, 1/\mathcal{C}_0)$ —monotonicity of $\frac{k_{\ell}(\omega)}{\omega}$.



Motivations O	Derivation OO	Characterization of Love waves	Recovering medium's parameters
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Weyl's law

$$\begin{array}{l} \text{Define } \bar{\nu}_j \equiv \bar{\nu}_j(y) := \frac{\nu_j(\omega, \omega y)}{\omega} = \sqrt{y^2 - C_j^{-2}}. \\ (n = 1) \ \text{For } y \in [1/\mathcal{C}_{\infty}, 1/\mathcal{C}_0), \ \mathcal{N}(\omega, y) \sim \frac{\omega}{\pi} |\tilde{\nu}_1(y)| \ \widetilde{T}_1 \ \text{as } \omega \to +\infty. \\ (n = 2) \ \text{For } y \in [1/\mathcal{C}_{\infty}, 1/\mathcal{C}_0), \ \text{as } \omega \to +\infty, \\ \\ \begin{cases} \mathcal{N}(\omega, y) \sim \frac{\omega}{\pi} |\tilde{\nu}_1(y)| \ \widetilde{T}_1, & \text{if } y \in [1/\widetilde{\mathcal{C}}_2, 1/\mathcal{C}_0), \\ \mathcal{N}(\omega, y) \sim \frac{\omega}{\pi} \left(|\tilde{\nu}_1(y)| \ \widetilde{T}_1 + |\tilde{\nu}_2(y)| \ \widetilde{T}_2 \right), & \text{if } y \in [1/\mathcal{C}_{\infty}, 1/\widetilde{\mathcal{C}}_2). \end{cases} \\ (n \geq 3) \ \text{For } y \in [1/\widetilde{\mathcal{C}}_2, 1/\mathcal{C}_0), \ \mathcal{N}(\omega, y) \sim \frac{\omega}{\pi} |\tilde{\nu}_1(y)| \ \widetilde{T}_1 \ \text{as } \omega \to +\infty. \end{array}$$

Motivations O	Derivation OO	Characterization of Love waves	Recovering medium's parameters
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Weyl's law

$$\begin{array}{l} \text{Define } \bar{\nu}_j \equiv \bar{\nu}_j(y) := \frac{\nu_j(\omega, \omega y)}{\omega} = \sqrt{y^2 - C_j^{-2}}. \\ (n = 1) \ \text{For } y \in [1/C_{\infty}, 1/C_0), \ N(\omega, y) \sim \frac{\omega}{\pi} |\tilde{\nu}_1(y)| \widetilde{T}_1 \ \text{as } \omega \to +\infty. \\ (n = 2) \ \text{For } y \in [1/C_{\infty}, 1/C_0), \ \text{as } \omega \to +\infty, \\ \\ \begin{cases} N(\omega, y) \sim \frac{\omega}{\pi} |\tilde{\nu}_1(y)| \widetilde{T}_1, & \text{if } y \in [1/\widetilde{C}_2, 1/C_0), \\ N(\omega, y) \sim \frac{\omega}{\pi} \left(|\tilde{\nu}_1(y)| \widetilde{T}_1 + |\tilde{\nu}_2(y)| \widetilde{T}_2 \right), & \text{if } y \in [1/C_{\infty}, 1/\widetilde{C}_2). \end{cases} \\ (n \geq 3) \ \text{For } y \in [1/\widetilde{C}_2, 1/C_0), \ N(\omega, y) \sim \frac{\omega}{\pi} |\tilde{\nu}_1(y)| \widetilde{T}_1 \ \text{as } \omega \to +\infty. \end{array}$$

Conjecture for $n \ge 3$

For
$$y \in [1/C_{\infty}, 1/C_0)$$
, as $\omega \to +\infty$,
 $N(\omega, y) \sim \frac{\omega}{\pi} \sum_{\rho=1}^{j} |\tilde{\nu}_{\rho}(y)| \widetilde{T}_{\rho}$, if $y \in [1/\widetilde{C}_{j+1}, 1/\widetilde{C}_{j})$.

Weyl's law

$$\begin{array}{l} \text{Define } \bar{\nu}_j \equiv \bar{\nu}_j(y) := \frac{\nu_j(\omega, \omega y)}{\omega} = \sqrt{y^2 - C_j^{-2}}. \\ (n = 1) \ \text{For } y \in [1/C_{\infty}, 1/C_0), \ N(\omega, y) \sim \frac{\omega}{\pi} |\tilde{\nu}_1(y)| \ \widetilde{T}_1 \ \text{as } \omega \to +\infty. \\ (n = 2) \ \text{For } y \in [1/C_{\infty}, 1/C_0), \ \text{as } \omega \to +\infty, \\ \\ \begin{cases} N(\omega, y) \sim \frac{\omega}{\pi} |\tilde{\nu}_1(y)| \ \widetilde{T}_1 \ , & \text{if } y \in [1/\widetilde{C}_2, 1/C_0) \ , \\ N(\omega, y) \sim \frac{\omega}{\pi} \left(|\tilde{\nu}_1(y)| \ \widetilde{T}_1 + |\tilde{\nu}_2(y)| \ \widetilde{T}_2 \right), & \text{if } y \in [1/C_{\infty}, 1/\widetilde{C}_2) \ . \end{cases} \\ (n \ge 3) \ \text{For } y \in [1/\widetilde{C}_2, 1/C_0), \ N(\omega, y) \sim \frac{\omega}{\pi} |\tilde{\nu}_1(y)| \ \widetilde{T}_1 \ \text{as } \omega \to +\infty. \end{array}$$

Recovering parameters:

- The "lines of accumulation" of branches are the $1/C_j = \sqrt{\rho_j/\mu_j}$. Hence we also know the functions $\tilde{\nu}_j$.
- The asymptotics give the thickness T_j .

Thank you for your attention!

Rayleigh waves



2 + 1-layers.