On the inverse problem for Love waves in a layered, elastic half-space

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Joint work with
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M. V. de Hoop, J. Garnier, A. Iantchenko, J. R., *Inverse problem for Love waves in a layered, elastic half-space*, e-print (2023), https://hal.science/hal-03994654

Motivations

Imaging crustal and upper mantle structures:
recovering the parameters of the medium
from the dispersion curves of the surface waves.

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Visual animations of surface waves [1]:

Love waves | Rayleigh waves

[1] Source: Seismological Facility for the Advancement of Geoscience (https://www.iris.edu/hq/inclass/animation/seismic_wave_motions4_waves_animated)

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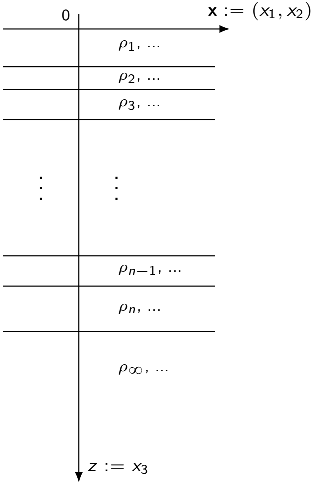

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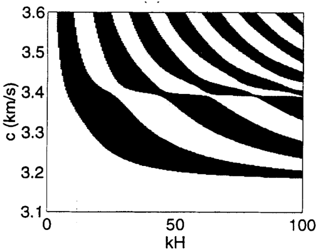
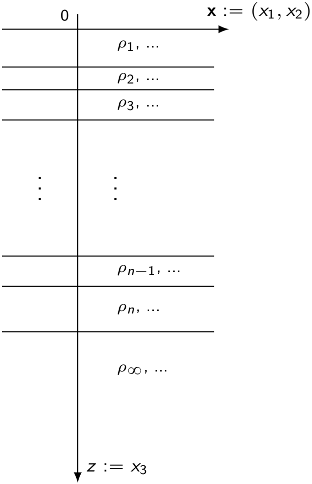


Figure from [BB-H96] (based on [Tho50] and [Has53]).



[BB-H96] Buchen & Ben-Hador. *Free-mode surface-wave computations* (1996). Geophys. J. Int.
[Tho50] Thomson. *Transmission of elastic waves through a stratified solid medium* (1950). J. Appl. Phys.
[Has53] Haskell. *The dispersion of surface waves on multilayered media* (1953). Bull. Seismol. Soc. Am.

Derivation of the equation from the elastic wave equation in $\mathbb{R} \times \mathbb{R}^2 \times [0, +\infty)$, w/o source term.

$$\begin{cases} \rho \partial_{tt} \mathbf{u} = \operatorname{div} \boldsymbol{\tau}(\mathbf{u}) & \text{on } \mathbb{R} \times \mathbb{R}^2 \times [0, +\infty)\,, & \text{(linear elastic wave equation)} \\ \boldsymbol{\tau}(\mathbf{u}) \cdot \mathbf{e}_3 = 0 & \text{at } z = 0\,. & \text{(stress-free (Neumann) BC)} \end{cases}$$

- displacement vectors $\mathbf{u}(t, \mathbf{x}, z) = (u_1(t, \mathbf{x}, z), u_2(t, \mathbf{x}, z), u_3(t, \mathbf{x}, z))$;
- mass density $\rho(t, \mathbf{x}, z)$;
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- mass density $\rho(t, \mathbf{x}, z)$;
- Cauchy stress tensor $\boldsymbol{\tau}(\mathbf{u})$, given by Hookes' law $\boldsymbol{\tau}(\mathbf{u}) = \mathbf{C}\boldsymbol{\varepsilon}(\mathbf{u})$:
  - stiffness tensor $\mathbf{C}(t, \mathbf{x}, z)$;
  - infinitesimal strain tensor $\boldsymbol{\varepsilon}$, given by the strain–displacement equation

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{\nabla \mathbf{u} + \nabla \mathbf{u}^{\mathsf{T}}}{2} \quad \Leftrightarrow \quad \varepsilon_{k\ell}(\mathbf{u}) = \frac{\partial_{x_k} u_\ell + \partial_{x_\ell} u_k}{2}\,.$$

$$\begin{cases} \rho \partial_{tt} \mathbf{u} = \operatorname{div}(\mathbf{C}\boldsymbol{\varepsilon}(\mathbf{u}))\,, \\ (\mathbf{C}\boldsymbol{\varepsilon}(\mathbf{u})) \cdot \mathbf{e}_3\big|_{z=0} = 0\,. \end{cases}$$

Derivation of the equation

Assumptions

- symmetries $C_{ijk\ell} = C_{jik\ell} = C_{k\ell ij}$ (standard, physical assumption).

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- symmetries $C_{ijk\ell} = C_{jik\ell} = C_{k\ell ij}$.
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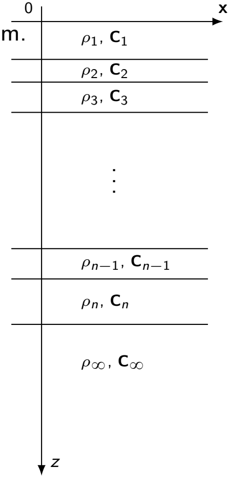

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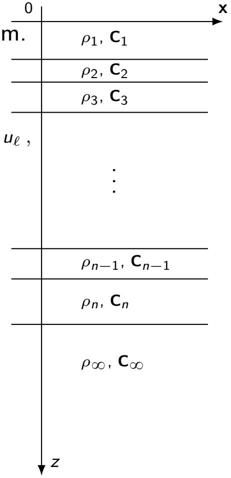
  $\Rightarrow \rho \equiv \rho(z), \mathbf{C} \equiv \mathbf{C}(z)$ and the i-th component reads

$$\rho \partial_{tt} u_i = \sum_{\ell=1}^{3} \left[\partial_z C_{i33\ell} \partial_z + \sum_{j=1}^{2} C_{ij3\ell} \partial_{x_j} \partial_z + \sum_{k=1}^{2} \partial_z C_{i3k\ell} \partial_{x_k} + \sum_{j,k=1}^{2} C_{ijk\ell} \partial_{x_j} \partial_{x_k} \right] u_\ell\,,$$



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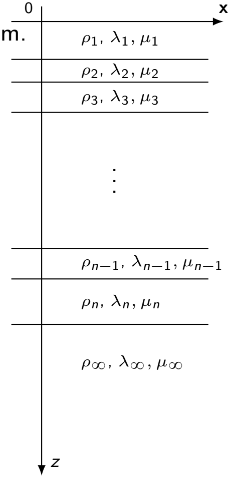
- symmetries $C_{ijk\ell} = C_{jik\ell} = C_{k\ell ij}$.
- stratified, homogeneous in (x, y)-plane, time-independent medium.
- isotropic media: $C_{ijk\ell} = \lambda \delta_i^j \delta_k^\ell + \mu \left(\delta_i^k \delta_j^\ell + \delta_i^\ell \delta_j^k \right)$
  with $\lambda \equiv \lambda(z)$ and $\mu \equiv \mu(z)$ the Lamé parameters.



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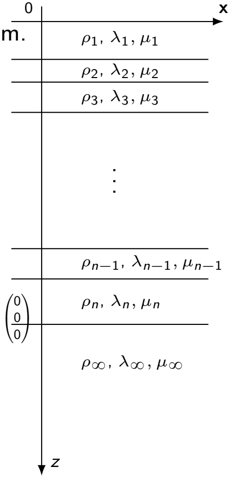

Defining the $(t, \mathbf{x})$-Fourier transform

$$\hat{u}_i(z) \equiv \hat{u}_i(\boldsymbol{\xi}, z, \omega) := \int_{\mathbb{R}^2}\int_{\mathbb{R}} u_i(\mathbf{x}, z, t)e^{\mathrm{i}\omega t}e^{\mathrm{i}\boldsymbol{\xi}\cdot\mathbf{x}}\,\mathrm{d}t\,\mathrm{d}\mathbf{x}$$

Then, $\rho\partial_{tt}\mathbf{u} = \mathrm{div}(\mathbf{C}\boldsymbol{\varepsilon}(\mathbf{u}))$ reads

$$\begin{pmatrix} (\lambda+\mu)\xi_1^2 + \mu|\boldsymbol{\xi}|^2 - \left(\partial_z\mu\partial_z + \rho\omega^2\right) & (\lambda+\mu)\xi_1\xi_2 & -\mathrm{i}\left(\lambda\partial_z + \partial_z\mu\right)\xi_1 \\ (\lambda+\mu)\xi_1\xi_2 & (\lambda+\mu)\xi_2^2 + \mu|\boldsymbol{\xi}|^2 - \left(\partial_z\mu\partial_z + \rho\omega^2\right) & -\mathrm{i}\left(\lambda\partial_z + \partial_z\mu\right)\xi_2 \\ -\mathrm{i}\left(\mu\partial_z + \partial_z\lambda\right)\xi_1 & -\mathrm{i}\left(\mu\partial_z + \partial_z\lambda\right)\xi_2 & \mu|\boldsymbol{\xi}|^2 - \left(\partial_z\left(\lambda + 2\mu\right)\partial_z + \rho\omega^2\right) \end{pmatrix}\begin{pmatrix}\hat{u}_1\\ \hat{u}_2\\ \hat{u}_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

and $(\mathbf{C}\boldsymbol{\varepsilon}(\mathbf{u}))\cdot\mathbf{e}_3\big|_{z=0} = 0$ reads

$$\begin{cases} \mathrm{i}\xi_1\hat{u}_3(0) + \partial_z\hat{u}_1(0) = 0\,, \\ \mathrm{i}\xi_2\hat{u}_3(0) + \partial_z\hat{u}_2(0) = 0\,, \\ \mathrm{i}\lambda(0)\left(\xi_1\hat{u}_1(0) + \xi_2\hat{u}_2(0)\right) + \left(\lambda(0) + 2\mu(0)\right)\partial_z\hat{u}_3(0) = 0\,. \end{cases}$$

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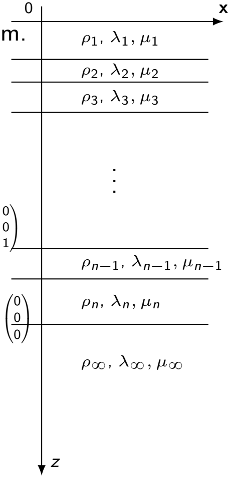

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and $(\phi_1, \phi_2, \phi_3)^{\mathsf{T}} := P(\boldsymbol{\xi})(\hat{u}_1, \hat{u}_2, \hat{u}_3)^{\mathsf{T}}$, with $P(\boldsymbol{\xi}) := \begin{pmatrix} \xi_2/|\boldsymbol{\xi}| & -\xi_1/|\boldsymbol{\xi}| & 0 \\ \xi_1/|\boldsymbol{\xi}| & \xi_2/|\boldsymbol{\xi}| & 0 \\ 0 & 0 & 1 \end{pmatrix}$

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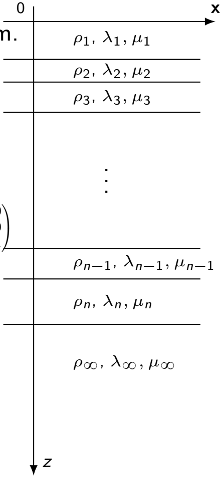

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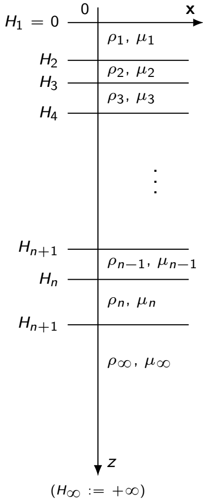
Love waves: definition (in our context)

$n + 1$ layers with constant shear modulus $\mu > 0$ and density $\rho > 0$.



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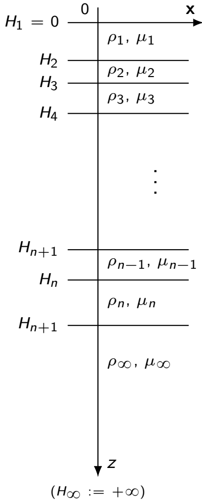
A Love wave *exists at* (ω, k) $:\Leftrightarrow \exists\, \phi \equiv \phi_{\omega,k} \in L^2((0,+\infty))\backslash\{0\}$ s.t.

$$\begin{cases} -\left(\mu\phi'\right)' = \mu\omega^2\left(\rho/\mu - k^2/\omega^2\right)\phi\,, \quad \text{on } [0,+\infty)\,, \\ \phi \in \mathcal{C}([0,+\infty))\,, \qquad \text{(continuity of displacement)} \\ \mu\phi' \in \mathcal{C}([0,+\infty)) \text{ with } \phi'(0) = 0\,. \quad \text{(continuity of shear and normal stress components)} \end{cases}$$



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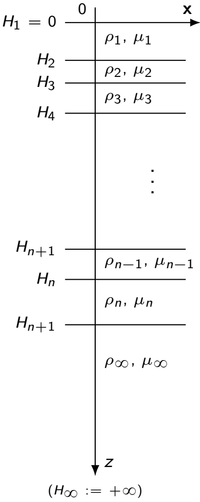
Defining $\nu_j \equiv \nu_j(\omega, k) := \omega\sqrt{k^2/\omega^2 - C_j^{-2}}$ (with $\operatorname{Im}\nu_j \leqslant 0$), where $C_j := \sqrt{\mu_j/\rho_j}$, then on each layer ϕ is either affine or of the form $A_{j,+}e^{+\nu_j z} + A_{j,-}e^{-\nu_j z}$.



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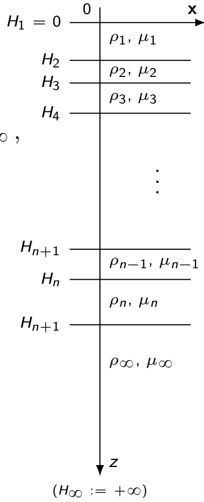

Love waves: characterization

Assuming for simplicity that (ω, k) is s.t. $\nu_j \neq 0$, if $\phi \equiv \phi_{\omega,k}$ exists, then

$$\phi(z) = \begin{cases} 2\alpha_1 \cosh[\nu_1 z]\,, & \text{if} \quad 0 \leqslant z < H_2\,, \\ \alpha_j e^{-\nu_j z} + \beta_j e^{+\nu_j z}\,, & \text{if} \quad H_j \leqslant z < H_{j+1}\,, \quad \forall\, j \in [\![2, n]\!]\,, \\ \alpha_{n+1} e^{-\nu_{n+1} z}\,, & \text{if} \quad H_{n+1} \leqslant z < +\infty\,. \end{cases}$$

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The $2n$ continuity conditions (on ϕ and $\mu\phi'$) at the boundaries $\{H_j\}_{2\leqslant j\leqslant n+1}$ yield

$$\begin{cases} 2\alpha_1 \cosh[\nu_1 H_2] = \beta_2 e^{+\nu_2 H_2} + \alpha_2 e^{-\nu_2 H_2}\,, \\ 2\mu_1\nu_1\alpha_1 \sinh[\nu_1 H_2] = \mu_2\nu_2 \left(\beta_2 e^{+\nu_2 H_2} - \alpha_2 e^{-\nu_2 H_2}\right), \\ \beta_{j-1} e^{+\nu_{j-1} H_j} + \alpha_{j-1} e^{-\nu_{j-1} H_j} = \beta_j e^{+\nu_j H_j} + \alpha_j e^{-\nu_j H_j}\,, & \forall\, j \in [\![3, n]\!]\,, \\ \mu_{j-1}\nu_{j-1} \left(\beta_{j-1} e^{+\nu_{j-1} H_j} - \alpha_{j-1} e^{-\nu_{j-1} H_j}\right) = \mu_j\nu_j \left(\beta_j e^{+\nu_j H_j} - \alpha_j e^{-\nu_j H_j}\right), & \forall\, j \in [\![3, n]\!]\,, \\ \beta_n e^{+\nu_n H_{n+1}} + \alpha_n e^{-\nu_n H_{n+1}} = \alpha_{n+1} e^{-\nu_{n+1} H_{n+1}}\,, \\ \mu_n\nu_n \left(\beta_n e^{+\nu_n H_{n+1}} - \alpha_n e^{-\nu_n H_{n+1}}\right) = -\mu_{n+1}\nu_{n+1}\alpha_{n+1} e^{-\nu_{n+1} H_{n+1}}\,. \end{cases}$$

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$$\begin{cases} 2\alpha_1 \cosh[\nu_1 H_2] = \beta_2 e^{+\nu_2 H_2} + \alpha_2 e^{-\nu_2 H_2}\,, \\ 2\mu_1\nu_1\alpha_1 \sinh[\nu_1 H_2] = \mu_2\nu_2 \left(\beta_2 e^{+\nu_2 H_2} - \alpha_2 e^{-\nu_2 H_2}\right), \\ \beta_{j-1} e^{+\nu_{j-1} H_j} + \alpha_{j-1} e^{-\nu_{j-1} H_j} = \beta_j e^{+\nu_j H_j} + \alpha_j e^{-\nu_j H_j}\,, & \forall\, j \in [\![3, n]\!]\,, \\ \mu_{j-1}\nu_{j-1} \left(\beta_{j-1} e^{+\nu_{j-1} H_j} - \alpha_{j-1} e^{-\nu_{j-1} H_j}\right) = \mu_j\nu_j \left(\beta_j e^{+\nu_j H_j} - \alpha_j e^{-\nu_j H_j}\right), & \forall\, j \in [\![3, n]\!]\,, \\ \beta_n e^{+\nu_n H_{n+1}} + \alpha_n e^{-\nu_n H_{n+1}} = \alpha_{n+1} e^{-\nu_{n+1} H_{n+1}}\,, \\ \mu_n\nu_n \left(\beta_n e^{+\nu_n H_{n+1}} - \alpha_n e^{-\nu_n H_{n+1}}\right) = -\mu_{n+1}\nu_{n+1}\alpha_{n+1} e^{-\nu_{n+1} H_{n+1}}\,. \end{cases}$$

A Love wave exists if there exists a non-zero solution $(\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_{n+1}, \beta_{n+1})$.

Love waves: characterization

I.e., if $D_n := \det \mathbb{M}_n = 0$, for

$$\mathbb{M}_n := \begin{pmatrix} L_1^r & R_2 & \mathbb{O}_2 & \mathbb{O}_2 & \mathbb{O}_2 & 0 \\ 0 & L_2 & R_3 & \ddots & \mathbb{O}_2 & \vdots \\ \vdots & \mathbb{O}_2 & \ddots & \ddots & \mathbb{O}_2 & \vdots \\ \vdots & \mathbb{O}_2 & \ddots & L_{n-1} & R_n & 0 \\ 0 & \mathbb{O}_2 & \mathbb{O}_2 & \mathbb{O}_2 & L_n & R_{n+1}^l \end{pmatrix}$$

where $L_1^r := 2\begin{pmatrix} \cosh[\nu_1 H_2] \\ \mu_1\nu_1 \sinh[\nu_1 H_2] \end{pmatrix}$, $R_{n+1}^l := \begin{pmatrix} -e^{-\nu_{n+1}H_{n+1}} \\ +\mu_{n+1}\nu_{n+1}e^{-\nu_{n+1}H_{n+1}} \end{pmatrix}$, and

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Why working with this big, complicated matrix instead of using the transfer matrix method (=propagator matrix method)?

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Why working with this big, complicated matrix instead of using the transfer matrix method (=propagator matrix method)?
↳ We read from it that there is at most one solution: the submatrix $\widetilde{\mathbb{M}}_n$ where the first row and the last column are removed, is block (upper) triangular with diagonal blocks $2\mu_1\nu_1 \sinh[\nu_1 H_2]$ and the L_j's, hence $\det \widetilde{\mathbb{M}}_n = 2^n \sinh[\nu_1 H_2] \prod_{j=1}^n \mu_j \nu_j \neq 0$.

Love waves: characterization

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where $L_1^r := 2\begin{pmatrix} \cosh[\nu_1 H_2] \\ \mu_1\nu_1 \sinh[\nu_1 H_2] \end{pmatrix}$, $R_{n+1}^l := \begin{pmatrix} -e^{-\nu_{n+1}H_{n+1}} \\ +\mu_{n+1}\nu_{n+1}e^{-\nu_{n+1}H_{n+1}} \end{pmatrix}$, and

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Why working with this big, complicated matrix instead of using the transfer matrix method (=propagator matrix method)?

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(It also holds if some ν_j's are zero.)

Love waves: dispersion relation

Proposition: dispersion relation

Let $n \in \mathbb{N}\backslash\{0\}$ and $T_j := H_{j+1} - H_j$, $j \in [\![1, n+1]\!]$, be the layers' thickness. Then,

$$\exists \text{ Love wave at } (\omega, k) \Leftrightarrow \begin{cases} f_n(\omega, k) := \mu_\infty \nu_\infty(\omega, k) P_n(\omega, k) + Q_n(\omega, k) = 0\,, \\ k \neq \omega / C_\infty\,, \end{cases}$$

where the P_n's and Q_n's are defined recursively by $P_0 = 1$, $Q_0 = 0$, and

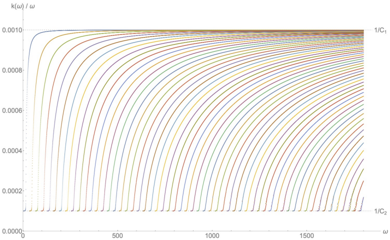
$$\begin{pmatrix} P_m \\ Q_m \end{pmatrix} = M_m \begin{pmatrix} P_{m-1} \\ Q_{m-1} \end{pmatrix} \quad \text{for all } m \in [\![1, n]\!]\,,$$

with

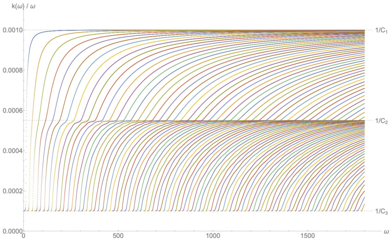
$$M_m := \begin{cases} \begin{pmatrix} \cosh[\nu_m T_m] & \sinh[\nu_m T_m]/(\mu_m \nu_m) \\ \mu_m \nu_m \sinh[\nu_m T_m] & \cosh[\nu_m T_m] \end{pmatrix} & \text{if } \nu_m \neq 0\,, \\ \begin{pmatrix} 1 & T_m/\mu_m \\ 0 & 1 \end{pmatrix} & \text{if } \nu_m = 0\,. \end{cases}$$

"$k \neq \omega/C_\infty$" can be replaced by "$\omega/C_\infty < k < \omega/C_0$" ($C_0 := \min_{[0,+\infty)} C$), because the solutions to $f_n(\omega, k) = 0$ are s.t. $k \in [\omega/C_\infty, \omega/C_0)$.

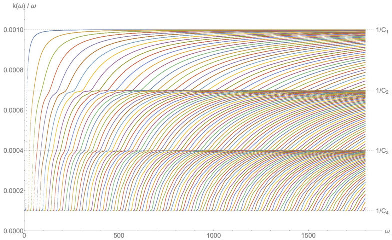
Love waves: numerics



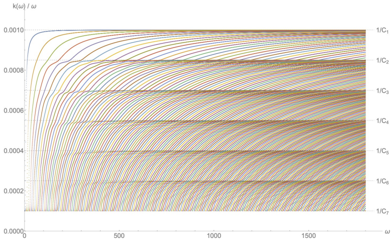
(1 + 1)-layers



(2 + 1)-layers



(3 + 1)-layers

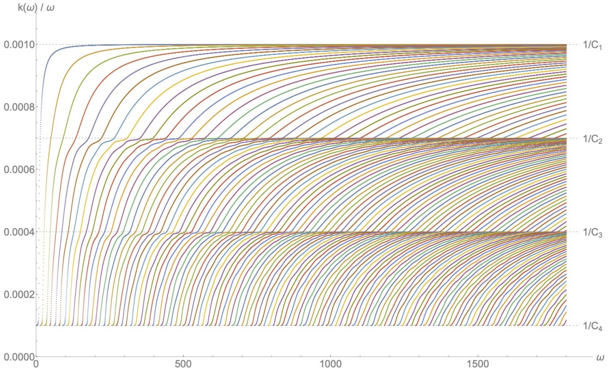


(6 + 1)-layers

Properties of the branches of k's

Definition–Lemma

Let $n \geqslant 1$. For any fixed $\omega > 0$, let the $k_\ell(\omega)$'s be the decreasingly ordered values $k \in \mathbb{R}\backslash\{\omega/C_\infty\}$ for which (ω, k) solves the dispersion relation $f_n(\omega, k) = 0$, i.e., a Love wave exists. Then, $k_\ell \in (\omega/C_\infty, \omega/C_0)$ and f_n is real-valued on $(0, +\infty) \times [\omega/C_\infty, \omega/C_0)$.



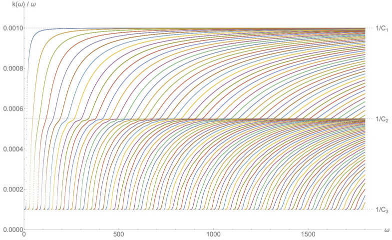
Properties of the branches of k's

Theorem

Let $n \geqslant 1$. For any ℓ, there exists $\omega_\ell \geqslant 0$ s.t.

$$(\omega_\ell, +\infty) \to (1/C_\infty, 1/C_0)$$
$$\omega \mapsto k_\ell(\omega)/\omega$$

is analytic, bijective, increasing.



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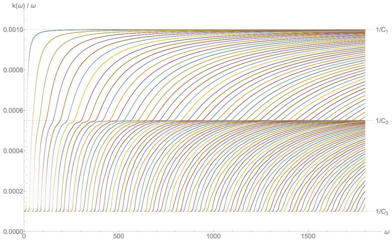

$\Rightarrow$ C_∞ and $C_0 := \min_j C_j$ are the inverse of the upper and lower limit values.

Properties of the branches of k's

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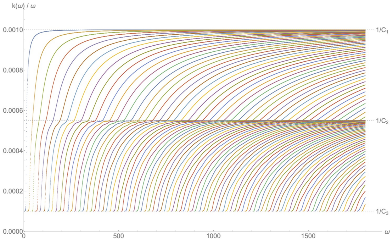

Idea of proof:

- ω fixed: $k_\ell(\omega)$'s in finite number, hence isolated.Uses the *simplicity* of the Love waves and complex analysis (Identity theorem).

Properties of the branches of k's

Theorem

Let $n \geqslant 1$. For any ℓ, there exists $\omega_\ell \geqslant 0$ s.t.

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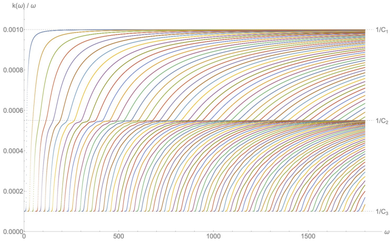

Idea of proof:

- ω fixed: $k_\ell(\omega)$'s in finite number, hence isolated.
- Branches $\omega \mapsto (k_\ell(\omega), \phi_\ell(\omega))$ exist on an open interval, with k_ℓ and ϕ_ℓ analytic. Uses by analytic perturbation theory (Kato–Rellich theorem), relying on the *simplicity* and *isolation*.

Properties of the branches of k's

Theorem

Let $n \geqslant 1$. For any ℓ, there exists $\omega_\ell \geqslant 0$ s.t.

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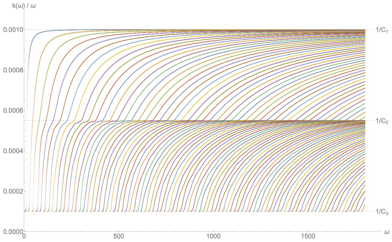

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- ω fixed: $k_\ell(\omega)$'s in finite number, hence isolated.
- Branches $\omega \mapsto (k_\ell(\omega), \phi_\ell(\omega))$ exist on an open interval, with k_ℓ and ϕ_ℓ analytic.
- Increasing: direct computation. Since $\omega \mapsto k_\ell(\omega)/\omega$ is differentiable.

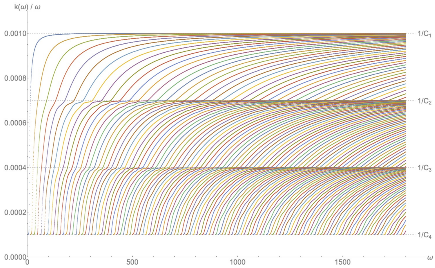
Weyl's law

Definition: number of branches $k_\ell(\omega)/\omega$ above or equal to y

Let $n \geqslant 1$, $\omega > 0$ and $y \in (1/C_\infty, 1/C_0)$. Define

$$N(\omega, y) := \#\left\{\ell \geqslant 1 \,:\, \frac{k_\ell(\omega)}{\omega} \geqslant y\right\}$$

with $k_\ell(\omega) = -\infty$ if k_ℓ is undefined at ω.



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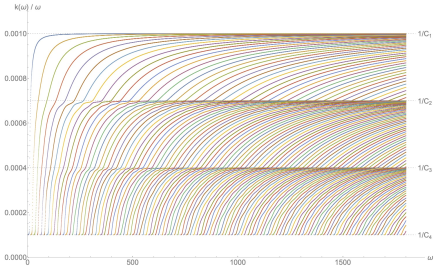
with $k_\ell(\omega) = -\infty$ if k_ℓ is undefined at ω.

Note: $\omega \mapsto N(\omega, y)$ is nondecreasing for $y \in (1/C_\infty, 1/C_0)$ —monotonicity of $\frac{k_\ell(\omega)}{\omega}$.



Weyl's law

Weyl's law

Define $\bar{\nu}_j \equiv \bar{\nu}_j(y) := \frac{\nu_j(\omega,\omega y)}{\omega} = \sqrt{y^2 - C_j^{-2}}$.

$(n = 1)$ For $y \in [1/C_\infty, 1/C_0)$, $N(\omega, y) \sim \frac{\omega}{\pi}|\tilde{\nu}_1(y)|\,\widetilde{T}_1$ as $\omega \to +\infty$.

$(n = 2)$ For $y \in [1/C_\infty, 1/C_0)$, as $\omega \to +\infty$,

$$\begin{cases} N(\omega, y) \sim \dfrac{\omega}{\pi}|\tilde{\nu}_1(y)|\,\widetilde{T}_1\,, & \text{if } y \in [1/\widetilde{C}_2, 1/C_0)\,, \\ N(\omega, y) \sim \dfrac{\omega}{\pi}\left(|\tilde{\nu}_1(y)|\,\widetilde{T}_1 + |\tilde{\nu}_2(y)|\,\widetilde{T}_2\right), & \text{if } y \in [1/C_\infty, 1/\widetilde{C}_2)\,. \end{cases}$$

$(n \geqslant 3)$ For $y \in [1/\widetilde{C}_2, 1/C_0)$, $N(\omega, y) \sim \frac{\omega}{\pi}|\tilde{\nu}_1(y)|\,\widetilde{T}_1$ as $\omega \to +\infty$.

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$(n \geqslant 3)$ For $y \in [1/\widetilde{C}_2, 1/C_0)$, $N(\omega, y) \sim \frac{\omega}{\pi}|\tilde{\nu}_1(y)|\,\widetilde{T}_1$ as $\omega \to +\infty$.

Conjecture for $n \geqslant 3$

For $y \in [1/C_\infty, 1/C_0)$, as $\omega \to +\infty$,

$$N(\omega, y) \sim \frac{\omega}{\pi}\sum_{p=1}^{j} |\tilde{\nu}_p(y)|\,\widetilde{T}_p\,, \qquad \text{if } y \in [1/\widetilde{C}_{j+1}, 1/\widetilde{C}_j)\,.$$

Weyl's law

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Define $\bar{\nu}_j \equiv \bar{\nu}_j(y) := \frac{\nu_j(\omega,\omega y)}{\omega} = \sqrt{y^2 - C_j^{-2}}$.

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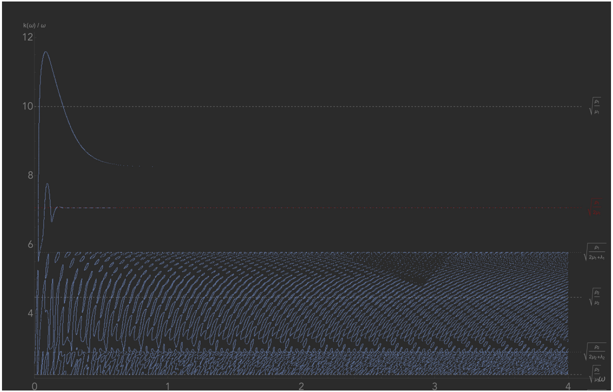
$(n \geqslant 3)$ For $y \in [1/\widetilde{C}_2, 1/C_0)$, $N(\omega, y) \sim \frac{\omega}{\pi}|\tilde{\nu}_1(y)|\,\widetilde{T}_1$ as $\omega \to +\infty$.

Recovering parameters:

- The "lines of accumulation" of branches are the $1/C_j = \sqrt{\rho_j/\mu_j}$. Hence we also know the functions $\tilde{\nu}_j$.
- The asymptotics give the thickness T_j.

Thank you for your attention!

Rayleigh waves



2 + 1-layers.