Inertia-gravity waves in geophysical vortices

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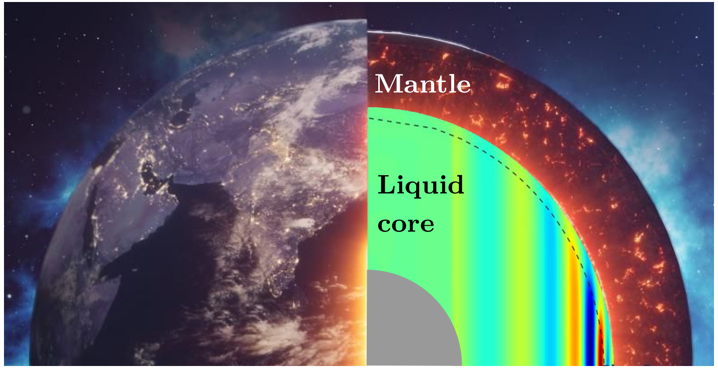
20-24 November 2023, Reims

Myself = Fluid dynamicist

Y. Colin de Verdière



+ **Astrophysical** flows
+ Lab. **experiments**

Series of papers (submitted)

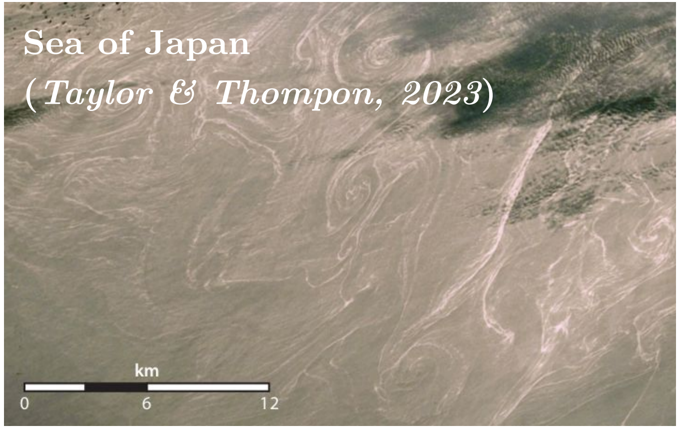
Colin de Verdière & **Vidal**, 2023, *The spectrum of the Poincaré operator in an ellipsoid*, **arXiv:2305.01369**
Vidal & Colin de Verdière, 2023, *Inertia-gravity waves in geophysical vortices*

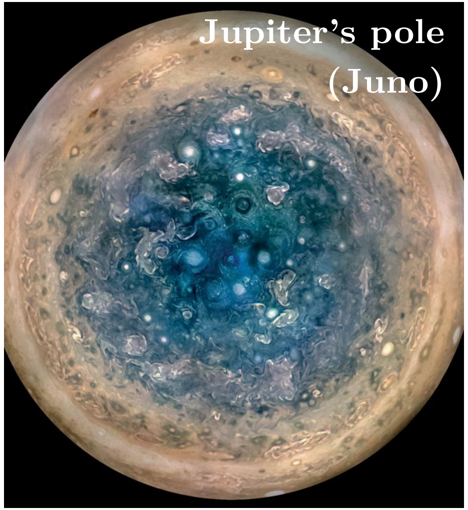
Vortex $\Rightarrow$ Vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{v} \neq \mathbf{0}$

Definition: **"What is [a vortex] ? It is like pornography. It is hard to defined but if you see it, you recongnise it immediately."**

G. K. Vallis

$<$ m | **$\sim$ km**





At all length scales!

INTRODUCTION

Standard equations for geophysical flows

$$\partial_t \boldsymbol{v} + 2\boldsymbol{\Omega} \times \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + \mathrm{d}_t \boldsymbol{\Omega} \times \boldsymbol{r} = -\nabla p + (\rho/\rho_*)\boldsymbol{g} + \nu \nabla^2 \boldsymbol{v} + \boldsymbol{f}$$

$$\nabla \cdot \boldsymbol{v} = 0$$

$$\partial_t \rho + \boldsymbol{v} \cdot \nabla \rho_0 = \kappa \nabla^2 \rho$$

Background density

Boussinesq equations

+ Boundary Conditions (**BC**)

2 key ingredients!

- Global **rotation**
- Density **stratification**

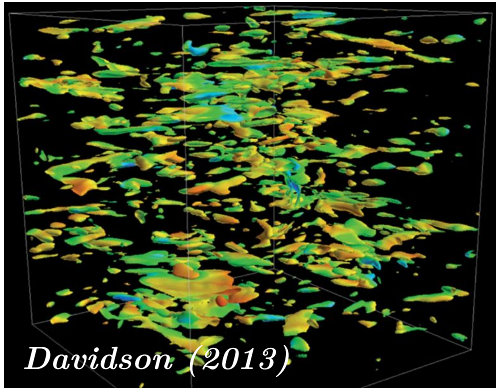
R >> S

Davidson (2013)

Elongated vortices

or

R << S



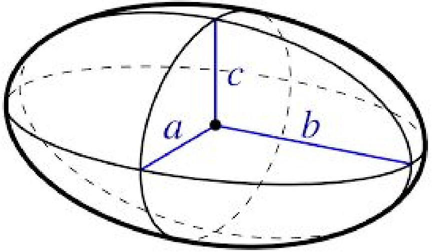
Davidson (2013)

Pancake-like vortices

INTRODUCTION

Dynamics of ellipsoidal vortices

- (Nearly) **isolated** vortices
- Fluid confined within an **ellipsoid**
- **Linear wave** motions

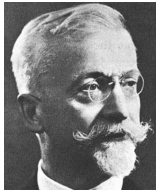


1800 - 1880

Dirichlet, Jacobi, Riemann...

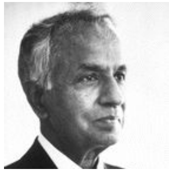
1880 - 1930

Poincaré



Cartan

1960 - 1990



Chandrasekhar
(Nobel prize)

Lebovitz

Friedlander

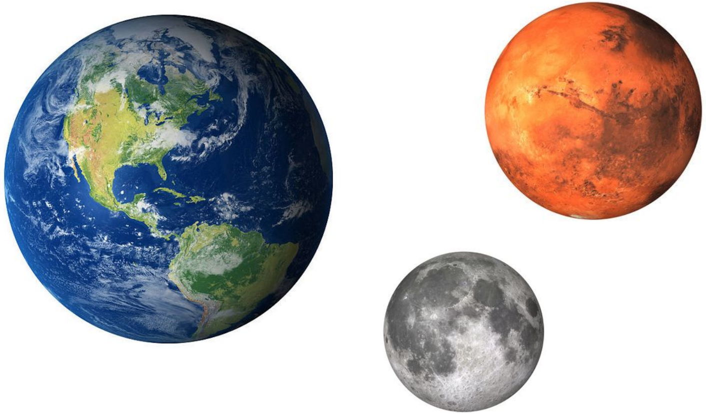
Long-standing history!

OUTLINE

2 **wave** families, 2 different **motivations** (but **similar** maths)

Paper I (CdV&Vidal)

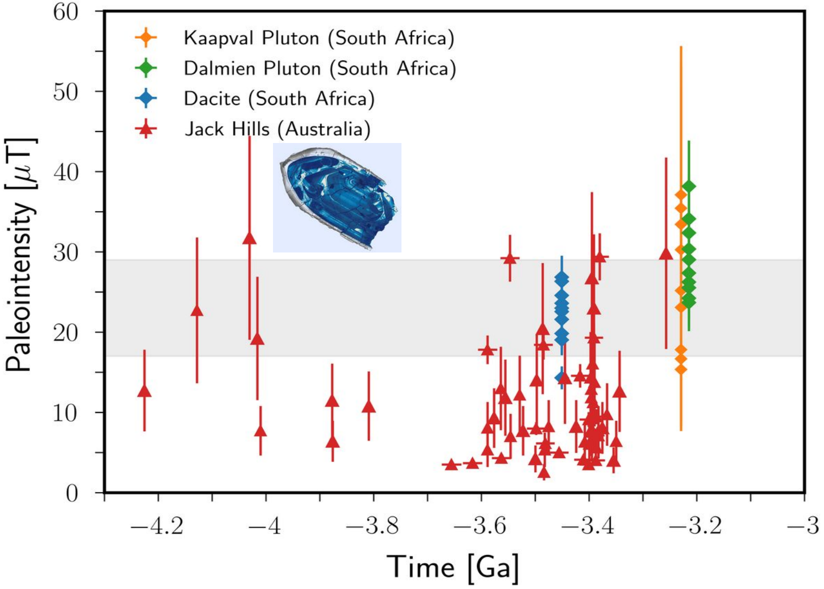
Inertial waves



Paper II (Vidal&CdV)

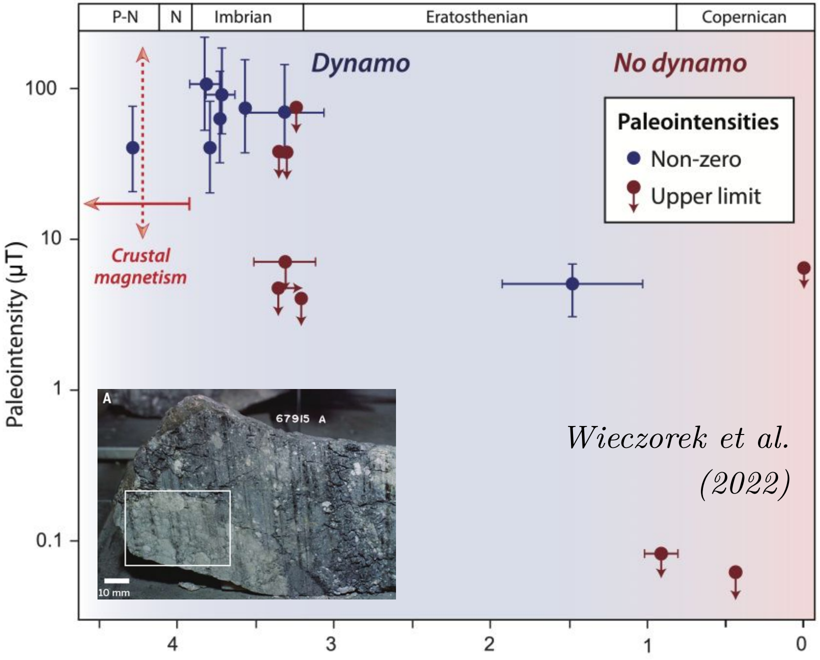
Inertia-gravity waves

Earth



Tarduno et al. (2015, 2020, 2023)
Borlina et al. (2020)

Moon



Wieczorek et al. (2022)

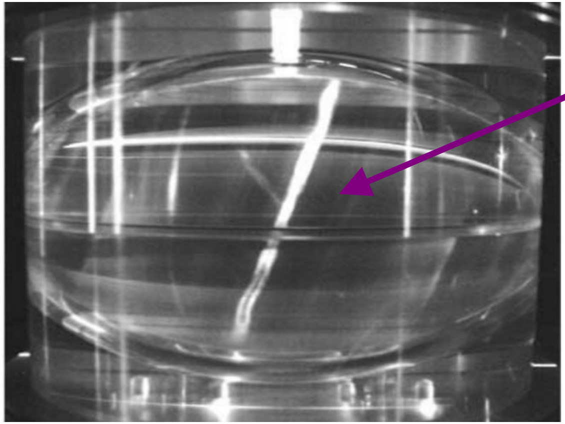
Origin of these dynamos: wave turbulence?

INERTIAL WAVES

Geophysical motivation (2/2)

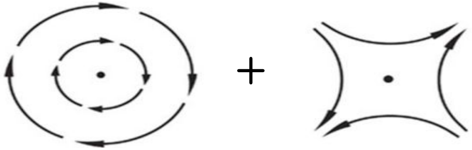
Orbitally driven (forced) **flow** → *Nonlinear interactions* $(\boldsymbol{v} \cdot \nabla)\boldsymbol{v}$ → **Turbulence**

Noir et al. (2003)



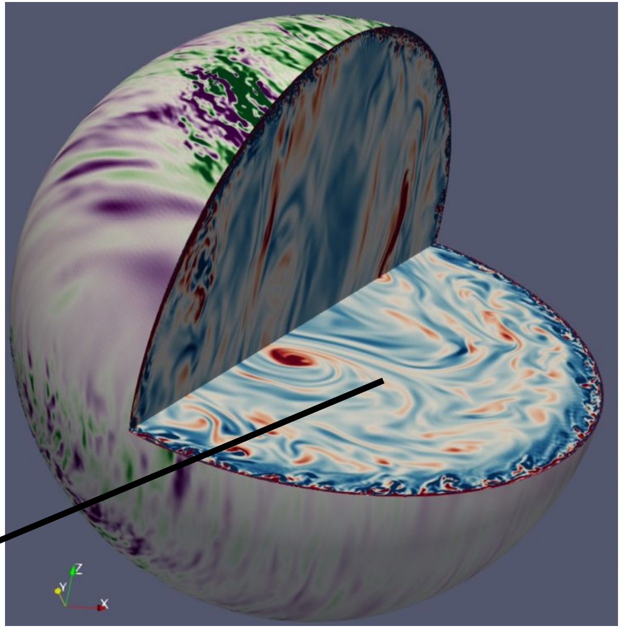
"**Poincaré**" flow

Rotation *Strain*

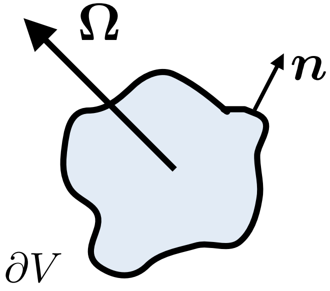


Sphere **Ellipsoid**

Cébron et al. (2019)



Wave turbulence



$$[\boldsymbol{v}, p] = [\boldsymbol{u}, \Phi] \exp(\mathrm{i}\omega t)$$

$$\mathrm{i}\omega \boldsymbol{u} + 2\boldsymbol{\Omega} \times \boldsymbol{u} = -\nabla\Phi$$
$$\nabla \cdot \boldsymbol{u} = 0$$
$$\boldsymbol{u} \cdot \boldsymbol{n}|_{\partial V} = 0$$

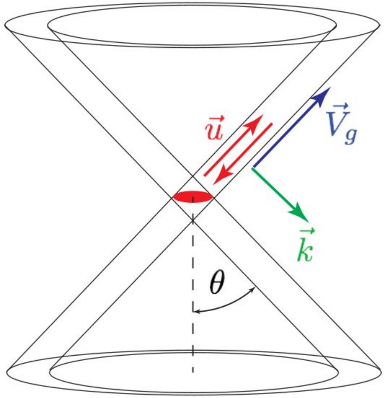
$\Longleftrightarrow$

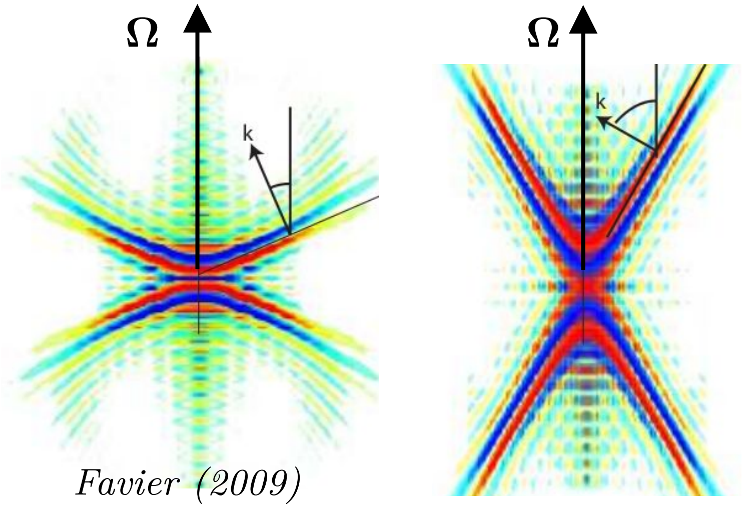
$$\omega^2 \nabla^2 \Phi = (2\boldsymbol{\Omega} \cdot \nabla)^2 \Phi.$$
$$\omega^2 \nabla\Phi \cdot \boldsymbol{n} = \mathrm{i}\omega (2\boldsymbol{\Omega} \times \boldsymbol{n}) \cdot \nabla\Phi + 4(\boldsymbol{\Omega} \cdot \nabla\Phi)(\boldsymbol{\Omega} \cdot \boldsymbol{n}) \text{ on } \partial V$$

Principal symbol

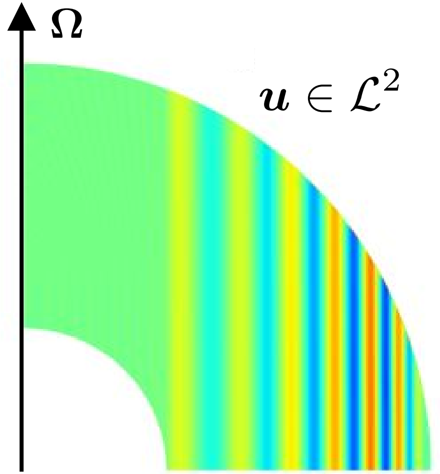
$$\mathfrak{p} := \omega^2 |\boldsymbol{k}|^2 - (2\boldsymbol{\Omega} \cdot \boldsymbol{k})^2$$

$$\begin{cases} |\omega| < 2|\boldsymbol{\Omega}| : & \text{Hyperbolic} \\ |\omega| > 2|\boldsymbol{\Omega}| : & \text{Elliptic} \end{cases}$$

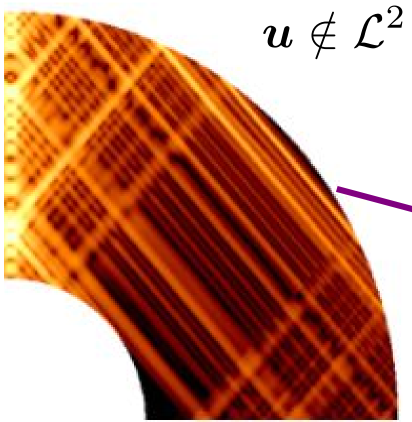




Favier (2009)



Vidal (2018)



Rieutord et al. (2001)

! Shell ≠ Ellipsoid !

An **ill-posed (self-adjoint)** problem

- (Almost empty) **point** spectrum
- **Continuous** spectrum

Attractors for Two-Dimensional Waves with Homogeneous Hamiltonians of Degree 0

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Ecole Normale Supérieure de Lyon, UMPA

Abstract

In domains with topography, inertial and internal waves exhibit interesting features. In particular, numerical and lab experiments show that, in two dimensions, for generic forcing frequencies, these waves concentrate on attractors. The goal of this paper is to analyze mathematically this behavior, using tools from spectral theory and microlocal analysis. © 2019 Wiley Periodicals, Inc.

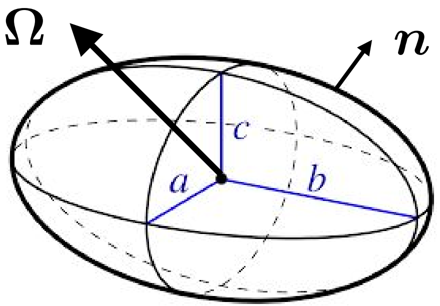
$\boldsymbol{\mathcal{V}}$: Hilbert space of square-integrable vector fields

$$\langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle := \int_V \boldsymbol{u}_1^\dagger \cdot \boldsymbol{u}_2 \, \mathrm{d}V$$

$$\boldsymbol{\mathcal{V}}^0 := \{\boldsymbol{u} \in \boldsymbol{\mathcal{V}}, \nabla \cdot \boldsymbol{u} = 0 \text{ in } V, \boldsymbol{u}\cdot\boldsymbol{n}|_{\partial V} = 0\}$$

$\boldsymbol{\mathcal{P}}_n$: Vector polynomial functions whose components $\propto x^i y^j z^k$ with $i + j + k \leq n$

$$\boldsymbol{\mathcal{V}}_n^0 := \boldsymbol{\mathcal{V}}^0 \cap \boldsymbol{\mathcal{P}}_n, \quad \dim(\boldsymbol{\mathcal{V}}_n^0) = n(n+1)(2n+7)/6$$



$$(\partial V) : \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$

Poincaré operator (bounded & self-adjoint)

$$\mathrm{i}\boldsymbol{\mathcal{C}}(\boldsymbol{u}) := \mathrm{i}\mathbb{L}(2\boldsymbol{\Omega} \times \boldsymbol{u}), \quad \mathbb{L} : \boldsymbol{\mathcal{V}} \to \boldsymbol{\mathcal{V}}^0$$

$$\left.\begin{aligned} &\boldsymbol{\mathcal{C}}|_{\boldsymbol{\mathcal{V}}_n^0} \subseteq \boldsymbol{\mathcal{V}}_n^0 \\ &\oplus_n \boldsymbol{\mathcal{V}}_n^0 \text{ dense in } \boldsymbol{\mathcal{V}}^0 \end{aligned}\right\}$$

Pure point spectrum in ellipsoids

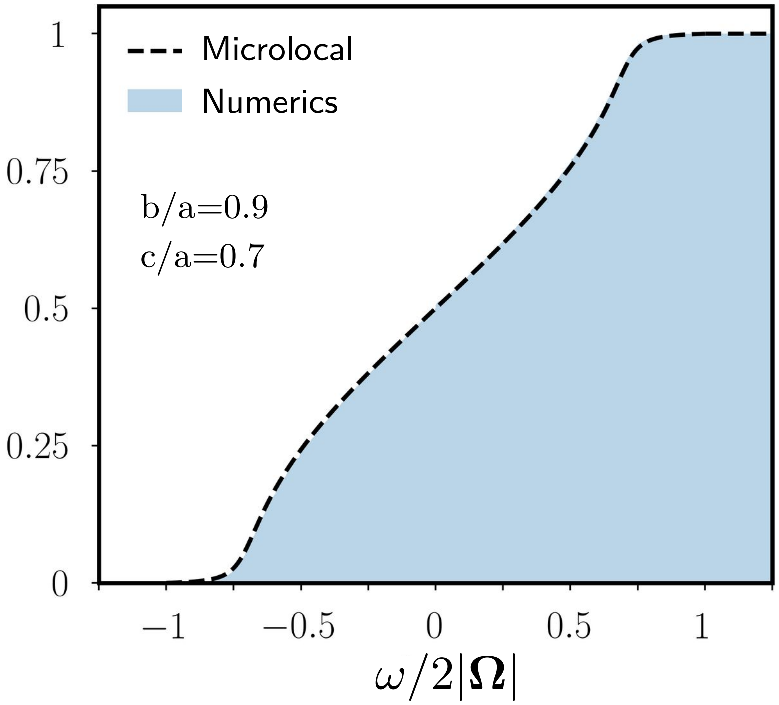
Asymptotic measure

$$\int_{-\infty}^{\lambda} \mathrm{d}\pi_\infty = \frac{1}{4\pi} \mathrm{Area}(\mathfrak{S}_\lambda \cap S^2)$$

$$\mathfrak{S}_\lambda := \{\tilde{\boldsymbol{k}} \in \mathbb{R}^3 \mid \omega(\tilde{\boldsymbol{k}}) \leq \lambda\}$$

$$\tilde{\boldsymbol{k}} = (k_x/a, k_y/b, k_z/c)^\top$$

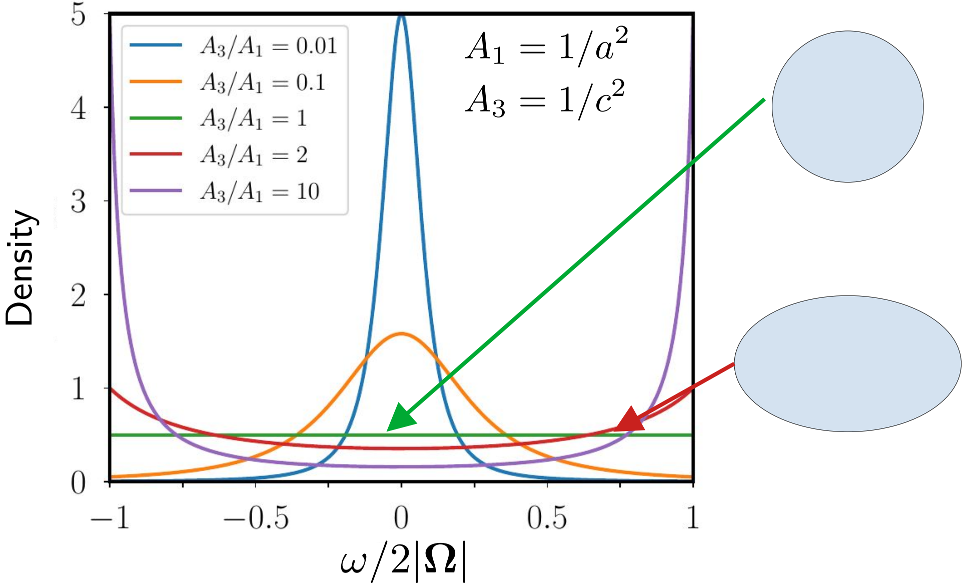
Rescaled **dispersion relation** of inertial waves



Proof:
- Pseudo-differential operator of degree 0
- Boutet de Monvel's calculus

...

Agreement with **numerics**



Expected for **planets** $(A_3/A_1 \sim 1)$

Geometry mismatch!

Experiments / Simulations $(A_3/A_1 \sim 3)$

Beyond ellipsoids?

- **Continuous** spectrum: attractors' properties?
- (Almost empty) **point** spectrum: **perturbation** theory?

OUTLINE

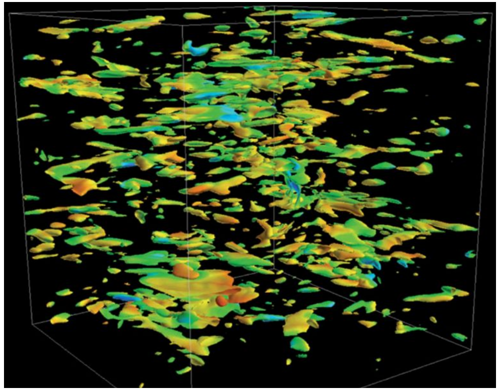
2 **wave** families, 2 different **motivations** (but **similar** maths)

Paper I (CdV&Vidal)

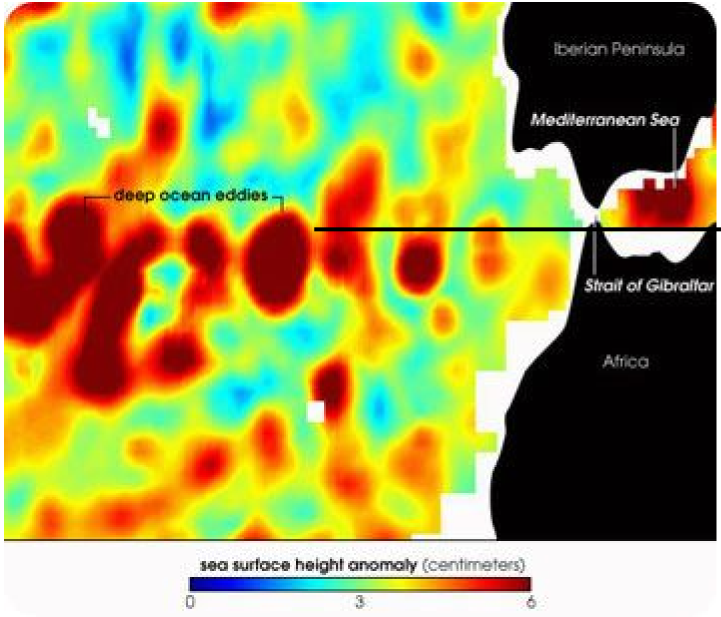
Inertial waves

Paper II (Vidal&CdV)

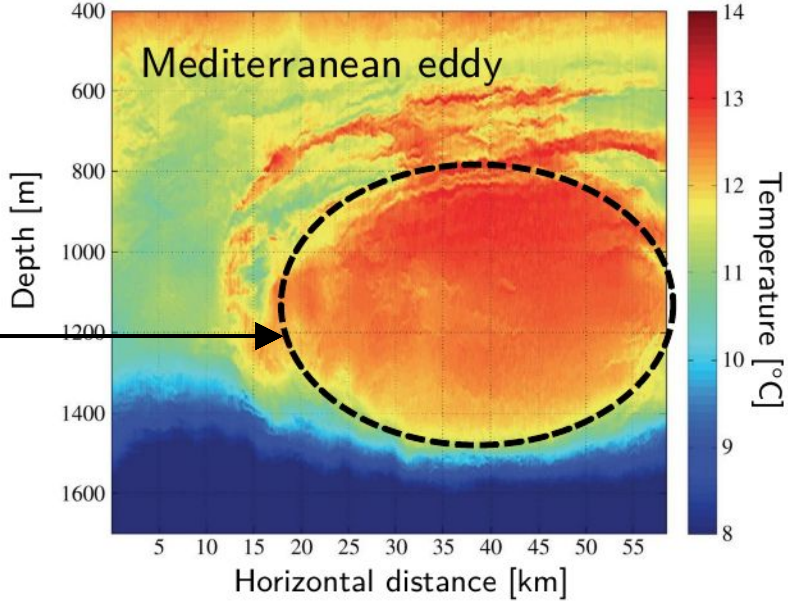
Inertia-gravity waves



‘‘Meddies’’

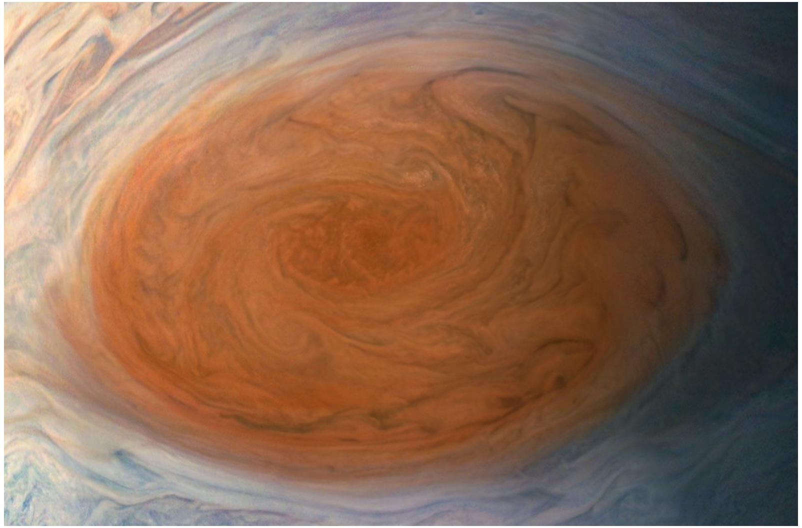


Yan et al. (2006)

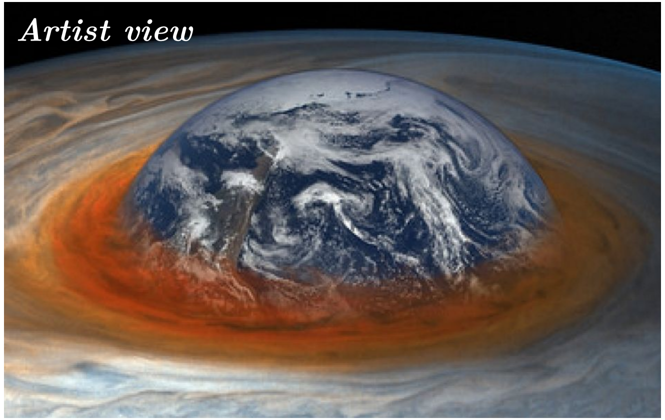


- **Anticylonic** vortices ($\rightarrow$ 50 cm.s^{-1})
- 10^9-10^{11} tons of **salt** per eddy

Jovian vortices (e.g. GRS)



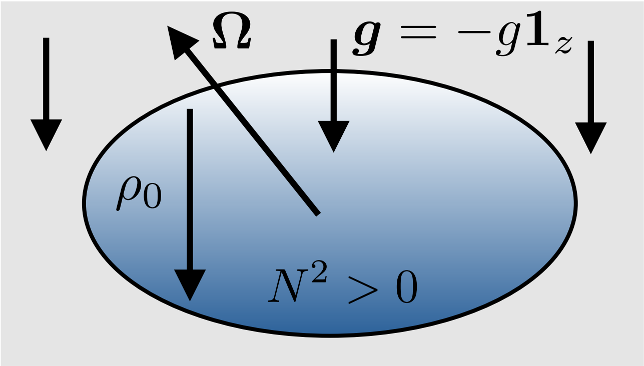
Juno - NASA - JPL-Caltech



- Horizontal: 10^3 - 10^4 km
- Vertical: a **few hundred** of km

Strongly flattened!

Nonlinear spectral problem in ellipsoids (1/3)



$$N^2 = \boldsymbol{g} \cdot \nabla \rho_0 / \rho_*$$

Brunt-Väisälä frequency

- $N^2 > 0$: Stably **stratified** (i.e. light above dense)
- $N^2 < 0$: Unstable

Primitive equations

$$\partial_t \boldsymbol{v} + 2\boldsymbol{\Omega} \times \boldsymbol{v} = -\nabla p + (\rho/\rho_*)\, \boldsymbol{g}$$

$$\nabla \cdot \boldsymbol{v} = 0$$

$$\partial_t \rho / \rho_* = (\boldsymbol{v} \cdot \mathbf{1}_z)\, N^2 / g$$

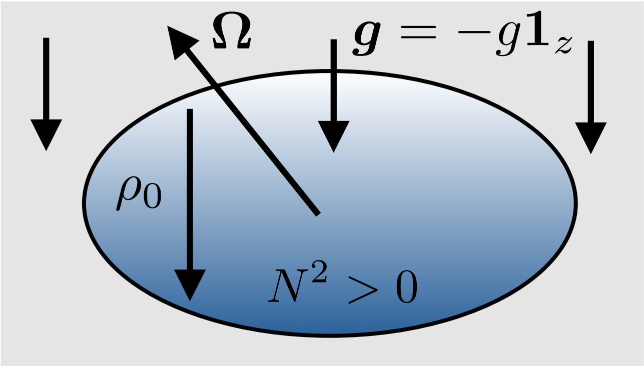
Wave-like equation

$$\partial_{tt}^2 \boldsymbol{v} + 2\boldsymbol{\Omega} \times (\partial_t \boldsymbol{v}) + N^2 (\boldsymbol{v} \cdot \mathbf{1}_z) \mathbf{1}_z = -\nabla \partial_t p$$

$$\Downarrow \quad [\boldsymbol{v}, p] = [\boldsymbol{u}, \Phi] \exp(\mathrm{i}\omega t)$$

$$-\omega^2 \boldsymbol{u} + 2\boldsymbol{\Omega} \times (\mathrm{i}\omega \boldsymbol{u}) + N^2 (\boldsymbol{u} \cdot \mathbf{1}_z) \mathbf{1}_z = -\mathrm{i}\omega \nabla \Phi$$

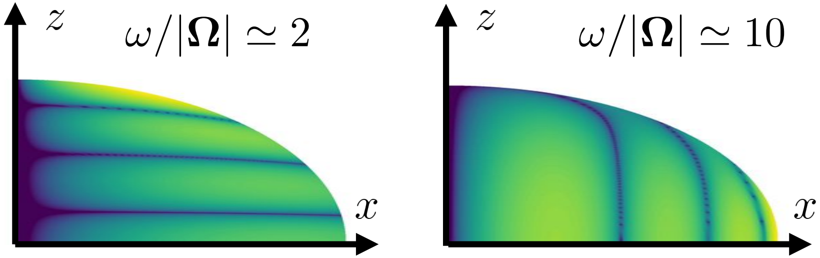
Nonlinear spectral problem in ellipsoids (2/3)



$$-\omega^2 \boldsymbol{u} + \omega \underbrace{\mathrm{i}\mathbb{L}(2\boldsymbol{\Omega} \times \boldsymbol{u})}_{\mathrm{i}\boldsymbol{\mathcal{C}}(\boldsymbol{u})} + \underbrace{\mathbb{L}(N^2(\boldsymbol{u} \cdot \mathbf{1}_z)\mathbf{1}_z)}_{\boldsymbol{\mathcal{K}}(\boldsymbol{u})} = \mathbf{0}$$

$$\left.\begin{aligned} &\boldsymbol{\mathcal{C}}|_{\boldsymbol{\mathcal{V}}_n^0} \subseteq \boldsymbol{\mathcal{V}}_n^0 \\ &\boldsymbol{\mathcal{K}}|_{\boldsymbol{\mathcal{V}}_n^0} \subseteq \boldsymbol{\mathcal{V}}_n^0 \\ &\oplus_n \boldsymbol{\mathcal{V}}_n^0 \text{ dense in } \boldsymbol{\mathcal{V}}^0 \end{aligned}\right\}$$ **Pure point** spectrum in ellipsoids

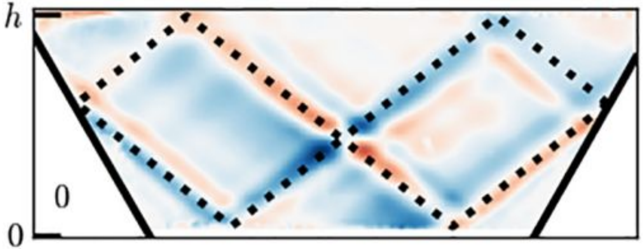
Square-integrable



$N/|\boldsymbol{\Omega}| = 20$

$\neq$

Attractors (e.g. in lab. experiment)



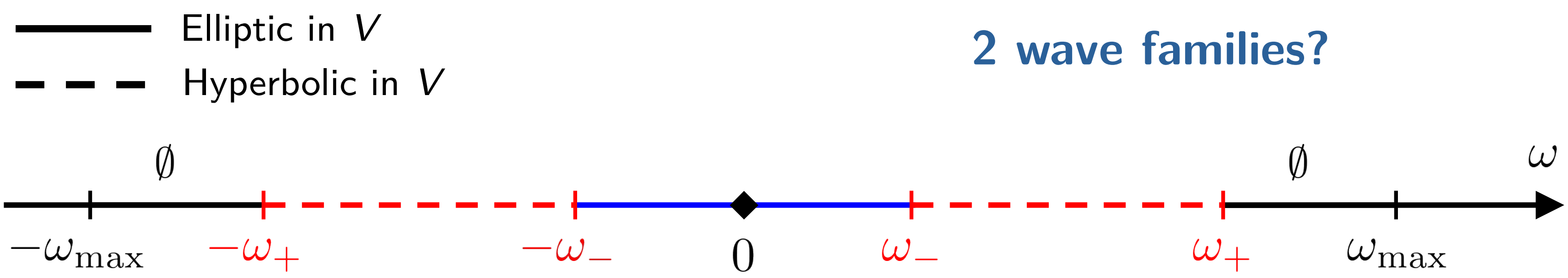
Pacary et al. (2023)

Principal symbol

$$\mathfrak{p} := |\boldsymbol{k}|^2\omega^2 - \left[N^2|\mathbf{1}_z \times \boldsymbol{k}|^2 + (2\boldsymbol{\Omega}\cdot\boldsymbol{k})^2\right]$$

$$\begin{cases} \omega_-^2 < \omega^2 < \omega_+^2 : & \text{Hyperbolic} \\ 0 < \omega^2 < \omega_-^2 : & \text{Elliptic} \end{cases}$$

$$2\omega_\pm^2 = \left[N^2 + 4|\boldsymbol{\Omega}|^2\right] \pm \sqrt{\left[N^2 + 4|\boldsymbol{\Omega}|^2\right]^2 - 16N^2(\boldsymbol{\Omega}\cdot\mathbf{1}_z)^2}$$



2 wave families?

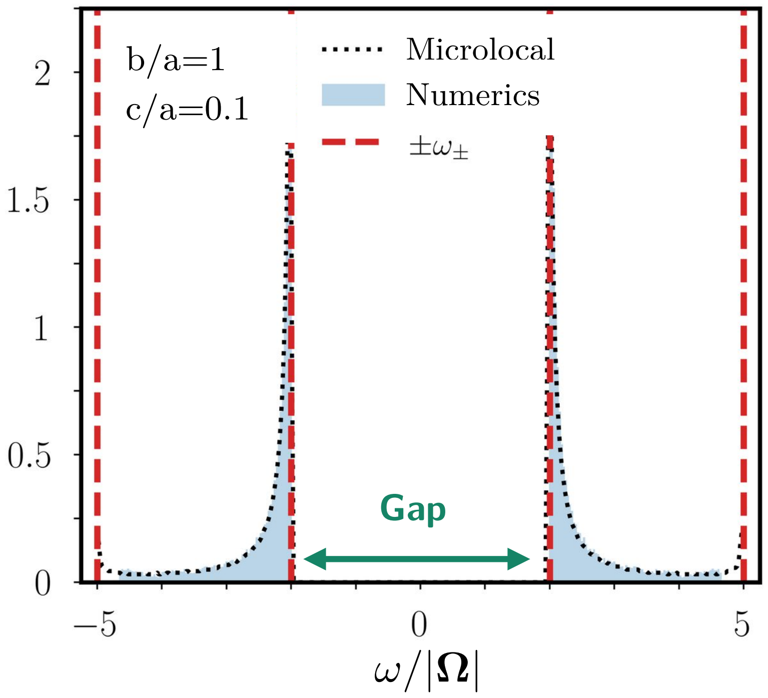
Asymptotic measure ($\omega_- < |\omega| < \omega_+$)

$$\int_{-\infty}^{\lambda} \mathrm{d}\pi_\infty = \frac{1}{8\pi}\mathrm{Area}(\mathfrak{S}_\lambda \cap S^2)$$

$$\mathfrak{S}_\lambda := \{\tilde{\boldsymbol{k}} \in \mathbb{R}^3 \mid \omega(\tilde{\boldsymbol{k}}) \leq \lambda\}$$

$$\tilde{\boldsymbol{k}} = (k_x/a, k_y/b, k_z/c)^\top$$

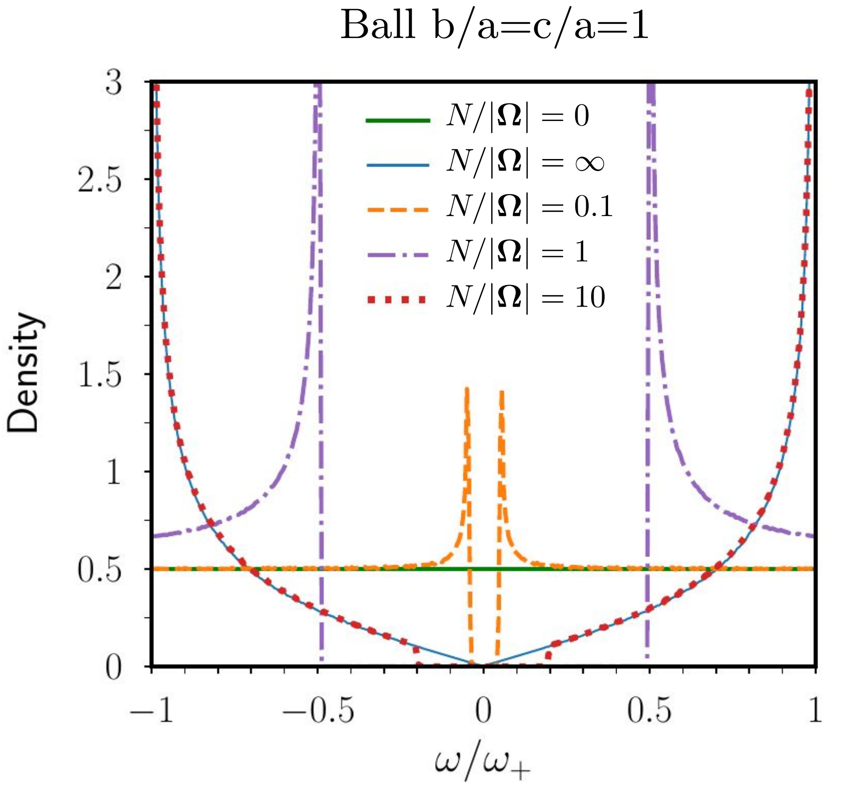
Rescaled **dispersion relation** of inertia-gravity waves



Proof: - Weyl asymptotics of the equivalent matrix operator

$$\boldsymbol{\mathcal{L}} := \begin{pmatrix} 0 & \boldsymbol{\mathcal{I}} \\ \boldsymbol{\mathcal{K}} & \mathrm{i}\boldsymbol{\mathcal{C}} \end{pmatrix}$$

Agreement with **numerics**



Some differences with pure inertial waves

- **Non-uniform** distribution in the ball
- Gap size function of the orientation of $\boldsymbol{\Omega}$

$$2\omega_\pm^2 = [N^2 + 4|\boldsymbol{\Omega}|^2] \pm \sqrt{[N^2+4|\boldsymbol{\Omega}|^2]^2 - 16N^2(\boldsymbol{\Omega}\cdot\mathbf{1}_z)^2}$$

- Different **orthogonality** conditions for $(\mathbf{u}_1, \mathbf{u}_2)$

$$\omega_1 \neq \omega_2$$

IW

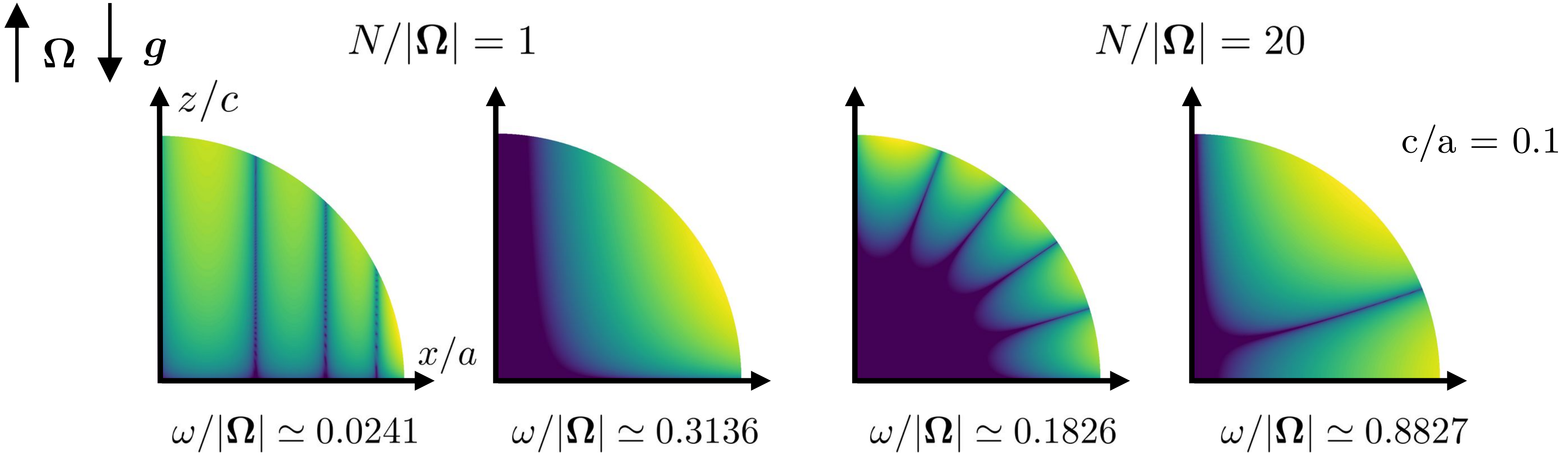
$$\langle \boldsymbol{u}_2, \boldsymbol{u}_1 \rangle = 0$$

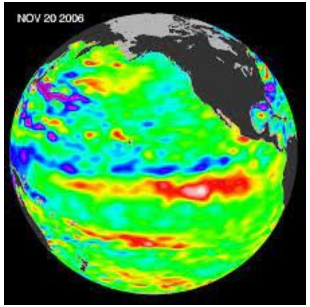
IGW

$$(\omega_2 + \omega_1)\langle \boldsymbol{u}_2, \boldsymbol{u}_1 \rangle = \langle \boldsymbol{u}_2, \mathrm{i}\boldsymbol{\mathcal{C}}(\boldsymbol{u}_1)\rangle$$

$$\omega_1\omega_2\langle \boldsymbol{u}_2, \boldsymbol{u}_1 \rangle = -\langle \boldsymbol{u}_2, \boldsymbol{\mathcal{K}}(\boldsymbol{u}_1)\rangle$$

Elliptic interval (1/2)





- **Similar** properties than (coastal & equatorial) **Kelvin waves**
- Dense **essential spectrum** when $0 < |\omega| < \omega_-$

Pressure problem

$$\mathcal{P}(\Phi) := -[\omega^2 - N^2]\nabla^2\Phi - N^2\left(\mathbf{1}_z \cdot \nabla\right)^2\Phi + \left(2\boldsymbol{\Omega}\cdot\nabla\right)^2\Phi = 0$$

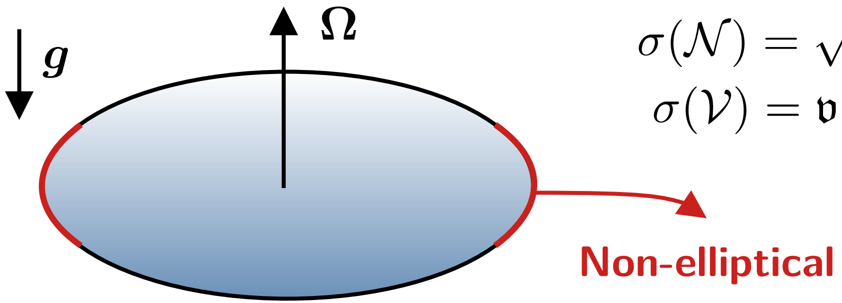
$$\boldsymbol{n}\cdot\nabla\Phi|_{\partial V} + \mathcal{B}(\boldsymbol{n}\times\nabla\Phi) = 0$$

- Principal symbol defines a **Riemannian** metric R on ∂V
- Unit normal vector **m** w/r R

Dirichlet-Neumann operator

$$\begin{aligned}\mathcal{P}(\Phi) &= 0\\ \boldsymbol{m}\cdot\nabla\Phi|_{\partial V} + \mathcal{V}(\Phi) &= 0\end{aligned} \quad = \quad \begin{aligned}\mathcal{N}(\Psi) &:= \boldsymbol{m}\cdot\nabla\Phi\\ \boxed{\mathcal{N}(\Psi) + \mathcal{V}(\Psi) = 0}\end{aligned} \quad + \quad \begin{aligned}\mathcal{P}(\Phi) &= 0\\ \Phi|_{\partial V} &= \Psi\end{aligned}$$

Principal symbol



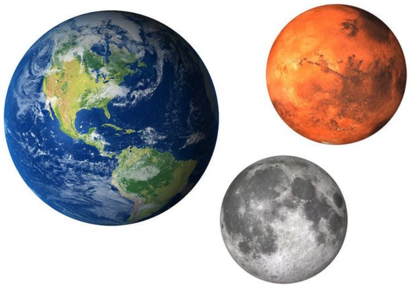
$$\left.\begin{aligned}\sigma(\mathcal{N}) &= \sqrt{\mathfrak{p}}\\ \sigma(\mathcal{V}) &= \mathfrak{v}\end{aligned}\right\} \quad (\sqrt{\mathfrak{p}} + \mathfrak{v})|_{\boldsymbol{k}\cdot\boldsymbol{m}=0} = 0$$

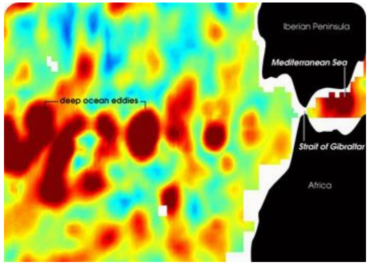
Shapiro-Lopatinskii condition

$\rightarrow$ **Essential spectrum!**

CONCLUSION & PERSPECTIVES

~ 0 ~ 10 ~ 100 $N/|\boldsymbol{\Omega}|$





Inertial waves

Inertia-gravity waves

Future papers in the series

- **Attractors** of Kelvin waves in other configurations?
- Kelvin waves with **radial gravity**?

Perspectives